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Space–Time Operators and the Incompatibility of Quantum Mechanics with Both Finite-Dimensional Spinor Fields and Lagrangian Dynamics in the Context of Special Relativity[1]

Robert Alan Coleman

Department of Physics, University of Toronto, Toronto, Ontario M5S 1A7, Canada

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It is shown that the present procedure used to quantize relativistic systems is inconsistent. Three mutually supporting arguments are given to sustain this conclusion. First, it is noted that the use of wave functions which transform according to any representation of 0(1, 3) whether finite or infinite dimensional is inappropriate because such a description allows for too many degrees of freedom. The phenomenon of Thomas precession indicates that internal structure such as spin and multipole moments must be described by mass shell (rest system) three-tensors rather than by unconstrained four-tensors. Second, even if representations of 0(1, 3) are employed, the momentum space construction for position-time operators, which is quite general and is applicable in any Euclidean or pseudo-Euclidean space, requires that the infinite-dimensional UIRs of 0(1, 3) be used rather than the finite-dimensional, nonunitary spinor representations. Third, various anomalous features of the customary kinematic formalism can be readily understood provided that this formalism is viewed as an *ad hoc* blend of two other formalisms which, while self-consistent, are incompatible except for the trivial case of free one-particle states. These criticisms focus attention on a number of specific weaknesses of the kinematic foundations of relativistic quantum mechanics and relativistic quantum field theory. These weaknesses are sufficiently serious to require a radical revision of the current theory even at the kinematic level.

1. INTRODUCTION

This paper presents an analysis of relativistic quantum kinematics. Although frequently regarded as trivial, kinematic considerations are

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extremely important because they form the descriptive foundation for more elaborate theoretical constructions. In this paper it is shown that the customary kinematic formalism which constitutes the present foundation of relativistic quantum mechanics and relativistic quantum field theory is internally inconsistent. The three basic arguments presented in support of this thesis are technical rather than philosophical in character. No constructive solution to these problems is offered. Indeed, the nature of the difficulties suggests that it may not be possible to construct a consistent relativistic quantum kinematics. At least, the resolution of the problems will require a radical departure from the currently accepted formalism.

The first argument is based on the construction of fully adequate position-time operators. It is generally accepted that an elementary system should be described by a wave function which transforms according to a unitary irreducible representation (UIR) of the Poincaré group $E(1, 3)$ (Wigner, 1939). This assumption leads to an elegant, canonical, momentum space description of elementary systems. Unfortunately, the corresponding position-time description of such elementary systems is not so firmly based. Consequently, the momentum space description is used for the construction of the position-time operators. Aside from the fact that any position-time operator involves some sort of momentum derivative, the construction used here bears little relation to previously published attempts (Newton and Wigner, 1949; Wightman and Schweber, 1955; Jordan and Mukunda, 1963; Fleming, 1965; Barut and Malin, 1968; Johnson, 1969; Broyles, 1970. Also, see references cited therein.) The construction is based on the observation that momentum space wave functions are intrinsic tensors on the mass hyperboloid, a space of constant negative curvature. This observation leads to a unique prescription for the momentum derivative as a covariant derivative with respect to the intrinsic geometry of the mass hyperboloid. The position-time operator is then defined in terms of this momentum derivative.

In Sections 2 and 3, the position-time operator is defined for the case of a particle of spin 1 with either a continuous or a discrete mass spectrum. This definition is then generalized to the case of arbitrary spin S in Section 4. The operator so defined is Hermitian, covariant, has appropriate commutation relations with the four-momentum operator, and is appropriately related to the angular momentum operator; however, the components of the position-time operator do not commute with each other. In Section 5, the relation of the position-time operator to the orbital and total angular momentum operators is studied in greater detail and it is shown that a position-time operator with components which mutually commute can be constructed only if infinite component wave functions which form a basis for a UIR of the homogeneous Lorentz group $0(1, 3)$ are used. (The spin zero case is an exception.) Thus in the context of relativistic quantum theory, *the existence of*

fully adequate position-time operators is incompatible with the use of finite-dimensional spinor fields. The construction used is available for any Euclidean or pseudo-Euclidean space regardless of signature. The parallel construction in the well-known case of three-dimensional Euclidean space is presented in Section 9 in order that the logic of the argument can be seen in a familiar context.

There are a number of anomalous features of the customary kinematic formalism of relativistic quantum mechanics which require a deeper analysis. As indicated above, one of these peculiarities is the use of finite-dimensional spinor fields which is incompatible with the existence of fully adequate position-time operators. That the customary formalism allows only the choice of Bose–Einstein statistics if infinite-component fields which transform under a UIR of $0(1, 3)$ are used suggests that the customary *theoretical* connection between spin and statistics is suspect. This suspicion is reinforced by the fact that no such connection obtains for the case of Schrödinger mechanics on three-dimensional Euclidean space. An additional problem is the fact that the position-time functional, even for a scalar Klein–Gordon field, does not transform properly under translations. These and other anomalies can be traced to the asymmetric treatment of the positive and negative frequency parts of the wave function, that is, to the reinterpretation and normal ordering principles. All of the anomalies can be readily understood provided that the customary kinematic formalism is viewed as an *ad hoc* blend of two other formalisms named the space–time density formalism and the flux density formalism.

In the space–time density formalism described in Section 6, an overall block space–time viewpoint is adopted in which events everywhere are described simultaneously. Note that a similar formalism may be constructed for any Euclidean or pseudo-Euclidean space regardless of signature. It is useful to compare the formulas of the space–time density formalism with their analogs in the familiar formalism used for Schrödinger mechanics in the Euclidean case. It is natural and essential to treat all momentum space star classes symmetrically. As a consequence of this symmetry, spin and statistics are decoupled; that is, either Bose–Einstein or Fermi–Dirac statistics may be used for any spin. The space–time density formalism has a natural positive-definite inner product and Hilbert space structure, features which are required for the probability interpretation of quantum mechanics. Unfortunately, the energy-momentum functional is indefinite.

The flux density formalism, described in Section 7, corresponds to the time-slice viewpoint in which space–time is viewed as the union of a family of spacelike hypersurfaces. This description is only appropriate for Minkowski-type spaces. The description is presented in terms of flux densities, of charge, energy momentum, and angular momentum, which determine the distribution

of various quantities on the spacelike hypersurfaces. If desired, these flux densities may also be regarded as densities with respect to the mass spectrum. It is interesting to note that there is no difficulty in defining a conserved position-time flux density the integral of which over a spacelike hypersurface yields the corresponding position-time functional. Again, it is natural to treat the positive and negative frequency parts of the wave function symmetrically, and again spin and statistics are *not* coupled. The flux density formalism does not have a positive-definite inner product. Instead, there is an indefinite sesquilinear form. However, the energy-momentum functional is positive definite. The bracket relations and conserved flux densities of this formalism are just the familiar ones encountered in classical Lagrangian field theory.

Neither the space–time density nor the flux density formalism has all of the features required for relativistic quantum mechanics. In the beginning of Section 8, it is shown that these two formalisms are compatible only for the trivial case of free, single-particle states, so that the difficulty cannot be resolved simply by using both formalisms together. The customary formalism represents an attempt to construct a workable system which, however, is only partially successful. The remainder of Section 8 is devoted to a discussion of the standard but *ad hoc* procedures used to give the flux density formalism a positive-definite inner product and a Hilbert space structure similar to that of the space–time density formalism. The resulting customary formalism has a number of peculiarities, including the fact that the position-time functional does not transform properly under space–time translations. This difficulty does not occur in either of the other two formalisms. The existence of such peculiarities indicates that the customary formalism is not internally consistent.

The third argument against the customary kinematic formalism is presented in Section 10. The history of a structureless, classical, point particle is represented by an everywhere timelike world line in Minkowski space. To describe a particle with structure, one must specify additional tensors at each point on the world line which vary smoothly with proper time. However, the phenomenon of Thomas precession shows that these tensors are not free but are constrained so that at each point of the world line the tensors are orthogonal to the tangent to the world line. Such constraints lead to difficulties in quantum mechanics because position and momentum cannot be known simultaneously with absolute precision. If the four momentum is unknown, then all quantities which describe features of internal structure, including spin and multipole moments, are also unknown. It is not appropriate to use wave functions which transform according to any representation of $0(1, 3)$ whether finite or infinite dimensional because the use of such representations ignores the presence of the geometric constraints and allows for more degrees

of freedom than actually exist. This additional freedom makes the choice of field somewhat arbitrary. Moreover, the constraints must be reimposed by means of additional equations of motion which seriously complicate the treatment of particles with higher spin. Such difficulties do not arise in the nonrelativistic case because time and space are separate so that one may simply use representations of 0(3) to describe three-dimensional objects in a three-dimensional space.

2. X^μ FOR SPIN 1 AND A CONTINUOUS MASS SPECTRUM

Since particle wave functions are intrinsic tensors on the mass hyperboloid [see Appendix A following equation (A.41)], the Hilbert space $\mathscr{H}$ for a spin 1 particle with a uniform, continuous mass spectrum is the set of complex-valued, mass hyperboloid vectors $\{\psi_a(M, \mathbf{v})\}$ square integrable with respect to the inner product

$$(\psi, \phi) = \int d\mu(M, \mathbf{v}) A^{ab}(\mathbf{v}) \psi_a^*(M, \mathbf{v}) \phi_b(M, \mathbf{v}) \tag{2.1}$$

where the measure $d\mu(M, \mathbf{v})$ is given by (A.17).

Given a tensor field ψ with components $\psi_a(M, \mathbf{v})$ and a Lorentz transformation Λ, define the tensor field $U(\Lambda)\psi$ with components $[U(\Lambda)\psi]_a(M, \mathbf{v})$ by

$$[U(\Lambda)\psi]_a(M, \mathbf{v}) = \frac{\partial(\Lambda^{-1}\mathbf{v})^b}{\partial v^a} \psi_b(M, \Lambda^{-1}\mathbf{v}) \tag{2.2}$$

The operators $\{U(\Lambda)\}$ form a representation of the Lorentz group since

$$\begin{aligned}
[U(\Lambda_1)\{U(\Lambda_2)\psi\}]_a(M, \mathbf{v}) &= \frac{\partial(\Lambda_1^{-1}\mathbf{v})^b}{\partial v^a} \{U(\Lambda_2)\psi\}_b(M, \Lambda_1^{-1}\mathbf{v}) \\
&= \frac{\partial(\Lambda_1^{-1}\mathbf{v})^b}{\partial v^a} \frac{\partial(\Lambda_2^{-1}\Lambda_1^{-1}\mathbf{v})^c}{\partial(\Lambda_1^{-1}\mathbf{v})^b} \psi_c(M, \Lambda_2^{-1}\Lambda_1^{-1}\mathbf{v}) \\
&= \frac{\partial((\Lambda_1\Lambda_2)^{-1}\mathbf{v})^b}{\partial v^a} \psi_b(M, (\Lambda_1\Lambda_2)^{-1}\mathbf{v}) \\
&= [U(\Lambda_1\Lambda_2)\psi]_a(M, \mathbf{v})
\end{aligned} \tag{2.3}$$

Moreover, this representation is unitary with respect to the inner product (2.1):

$$
\begin{aligned}
&(U(\Lambda)\psi,\ U(\Lambda)\phi) \\
&\quad = \int d\mu(M, \mathbf{v}) A^{ab}(\mathbf{v})[U(\Lambda)\psi]_a^*(M, \mathbf{v})[U(\Lambda)\phi]_b(M, \mathbf{v}) \\
&\quad = \int d\mu(M, \mathbf{v}) A^{ab}(\mathbf{v}) \frac{\partial(\Lambda^{-1}\mathbf{v})^c}{\partial v^a} \psi_c^*(M, \Lambda^{-1}\mathbf{v}) \frac{\partial(\Lambda^{-1}\mathbf{v})^d}{\partial v^b} \phi_d(M, \Lambda^{-1}\mathbf{v}) \\
&\quad = \int d\mu(M, \Lambda^{-1}\mathbf{v}) A^{ab}(\Lambda^{-1}\mathbf{v}) \psi_a^*(M, \Lambda^{-1}\mathbf{v}) \phi_b(M, \Lambda^{-1}\mathbf{v}) \\
&\quad = (\psi, \phi)
\end{aligned}
\tag{2.4}
$$

where the transformation property of the metric tensor $A^{ab}(\mathbf{v})$ and the fact that $d\mu$ is an invariant measure have been used.

In this Hilbert space, the position-time operator is represented by the momentum energy gradient given by (A.38). Since the wave functions are mass hyperboloid vectors rather than scalars, the partial derivative $\partial/\partial v^b$ is replaced by the covariant derivative $\delta/\delta v^b$. Thus

$$
\begin{aligned}
(\psi, X^\mu\phi) = i \int d\mu(M, \mathbf{v}) \Bigg[& A^{ab}(\mathbf{v}) \psi_a^*(M, \mathbf{v}) u^\mu(\mathbf{v}) \frac{\partial \phi_b}{\partial M}(M, \mathbf{v}) \\
& - \frac{1}{M} A^{ab}(\mathbf{v}) A^{cd}(\mathbf{v}) \psi_a^*(M, \mathbf{v}) u^\mu_{,c}(\mathbf{v}) \frac{\delta \phi_b(M, \mathbf{v})}{\delta v^d} \Bigg]
\end{aligned}
\tag{2.5}
$$

and

$$
\begin{aligned}
(X^\mu\psi, \phi) = -i \int d\mu(M, \mathbf{v}) \Bigg[& A^{ab}(\mathbf{v}) \frac{\partial \psi_a^*(M, \mathbf{v})}{\partial M} u^\mu(\mathbf{v}) \phi_b(M, \mathbf{v}) \\
& - \frac{1}{M} A^{ab}(\mathbf{v}) A^{cd}(\mathbf{v}) u^\mu_{,c}(\mathbf{v}) \frac{\delta \psi_a^*(M, \mathbf{v})}{\delta v^d} \phi_b(M, \mathbf{v}) \Bigg]
\end{aligned}
\tag{2.6}
$$

The operator X^μ is Hermitian provided (2.5) and (2.6) are equal. This demonstration is carried out in Appendix B. It is clear from the covariant structure of X^μ that the matrix elements transform as a Minkowski four vector under the Lorentz transformations defined by (2.2); however, for completeness, an explicit demonstration is given in Appendix C.

For convenience, write the operator X^μ in the symbolic form

$$
X^\mu = i \left[u^\mu(\mathbf{v}) \frac{\partial}{\partial M} - \frac{1}{M} A^{cd}(\mathbf{v}) u^\mu_{,c}(\mathbf{v}) \frac{\delta}{\delta v^d} \right] \tag{2.7}
$$

Then

$$
\begin{aligned}
X^\mu X^\nu = (i)^2 &\left[u^\mu(\mathbf{v})\frac{\partial}{\partial M} - \frac{1}{M}A^{ab}(\mathbf{v})u^\mu_{,a}(\mathbf{v})\frac{\delta}{\delta v^b}\right] \\
&\times \left[u^\nu(\mathbf{v})\frac{\partial}{\partial M} - \frac{1}{M}A^{cd}(\mathbf{v})u^\nu_{,c}(\mathbf{v})\frac{\delta}{\delta v^d}\right] \\
= -&\left[u^\mu(\mathbf{v})u^\nu(\mathbf{v})\frac{\partial^2}{\partial M^2} + \frac{1}{M^2}u^\mu(\mathbf{v})A^{cd}(\mathbf{v})u^\nu_{,c}(\mathbf{v})\frac{\delta}{\delta v^d}\right. \\
&- \frac{1}{M}u^\mu(\mathbf{v})A^{cd}(\mathbf{v})u^\nu_{,c}\frac{\delta}{\delta v^d}\frac{\partial}{\partial M} - \frac{1}{M}u^\nu(\mathbf{v})A^{ab}(\mathbf{v})u^\mu_{,a}(\mathbf{v})\frac{\delta}{\delta v^b}\frac{\partial}{\partial M} \\
&- \frac{1}{M}A^{ab}(\mathbf{v})u^\mu_{,a}(\mathbf{v})u^\nu_{,b}(\mathbf{v})\frac{\partial}{\partial M} + \frac{1}{M^2}A^{ab}(\mathbf{v})A^{cd}(\mathbf{v})u^\mu_{,a}(\mathbf{v})u^\nu_{,c}(\mathbf{v})\frac{\delta^2}{\delta v^b \delta v^d} \\
&\left. + \frac{1}{M^2}A^{ab}(\mathbf{v})A^{cd}(\mathbf{v})u^\mu_{,a}(v)\frac{\delta u^\nu_{,c}}{\delta v^b}\frac{\delta}{\delta v^d}\right] \qquad (2.8)
\end{aligned}
$$

Using (A.37) in the last term, one finds that, except for the term containing the second-order covariant derivative, $X^\mu X^\nu$ is symmetric in μ and ν. One obtains

$$
[X^\mu, X^\nu]_{ab} = \frac{1}{M^2}[u^\mu_{,a}(\mathbf{v})u^\nu_{,b}(\mathbf{v}) - u^\nu_{,a}(\mathbf{v})u^\mu_{,b}(\mathbf{v})] \qquad (2.9)
$$

The momentum-energy operator is just the multiplicative operator

$$
P^\mu = Mu^\mu(\mathbf{v}) \qquad (2.10)
$$

One readily obtains

$$
[X^\mu, P^\nu] = i[u^\mu(\mathbf{v})u^\nu(\mathbf{v}) - A^{ab}(\mathbf{v})u^\mu_{,a}(\mathbf{v})u^\nu_{,b}(\mathbf{v})] \qquad (2.11)
$$

or

$$
[X^\mu, P^\nu] = ig^{\mu\nu} \qquad (2.12)
$$

Define the orbital angular momentum by

$$
L^{\mu\nu} = X^\mu P^\nu - X^\nu P^\mu \qquad (2.13)
$$

Since the operators X^μ and P^μ are covariant and Hermitian, the operator $L^{\mu\nu}$ is as well. It follows from the commutation relations (2.9) and (2.12), that

$$
\begin{aligned}
[L^{\mu\nu}, L^{\rho\sigma}] = &\, i[g^{\mu\rho}L^{\nu\sigma} + g^{\nu\sigma}L^{\mu\rho} - g^{\mu\sigma}L^{\nu\rho} - g^{\nu\rho}L^{\mu\sigma}] \\
&- i[u^\mu(\mathbf{v})u^\rho(\mathbf{v})S^{\nu\rho} + u^\nu(\mathbf{v})u^\sigma(\mathbf{v})S^{\mu\rho} \\
&- u^\mu(\mathbf{v})u^\sigma(\mathbf{v})S^{\nu\rho} - u^\nu(\mathbf{v})u^\rho(\mathbf{v})S^{\mu\sigma}] \qquad (2.14)
\end{aligned}
$$

where

$$
S^{\mu\nu}_{ab} = i[u^\mu_{,a}(\mathbf{v})u^\nu_{,b}(\mathbf{v}) - u^\nu_{,a}(\mathbf{v})u^\mu_{,b}(\mathbf{v})] \qquad (2.15)
$$

The explicit form of $L^{\mu\nu}$ is

$$L^{\mu\nu} = iA^{ab}(\mathbf{v})[u^{\mu}(\mathbf{v})u^{\nu}_{,a}(\mathbf{v}) - u^{\nu}(\mathbf{v})u^{\mu}_{,a}(\mathbf{v})]\frac{\delta}{\delta v^b} \tag{2.16}$$

This operator is independent of M and consequently can be used even in the case of a discrete mass spectrum.

The commutation relations (2.9) and (2.14) are not those desired for the position-time and orbital angular momentum operators. The problem arises from restricting attention to wave functions describing a *single* spin. The problem is further discussed and resolved in Section 5.

3. X^{μ} FOR SPIN 1 AND A DISCRETE MASS SPECTRUM

For a particle with spin 1 and a given fixed mass M, the Hilbert space $\mathscr{H}$ is the set of complex-valued, mass hyperboloid vectors $\{\psi_a(\mathbf{v})\}$ square integrable with respect to the inner product

$$(\psi, \phi) = \frac{M^2}{2}\int d\mu(\mathbf{v})A^{ab}(\mathbf{v})\psi^*_a(\mathbf{v})\phi_b(\mathbf{v}) \tag{3.1}$$

The factor $M^2/2$ has been chosen in order to agree with the usual normalization. The representation of the Lorentz group given by (2.2) is not affected by this change in the Hilbert space.

The expression for the position-time operator must be changed since the mass is now a constant. Define

$$(\psi, X^{\mu}\phi) = i\frac{M^2}{2}\int d\mu(\mathbf{v})\left[-\frac{1}{M}A^{ab}(\mathbf{v})A^{cd}(\mathbf{v})\psi^*_a(\mathbf{v})u^{\mu}_{,c}(v)\frac{\delta\phi_b(\mathbf{v})}{\delta v^d}\right.$$
$$\left. - \frac{3}{2M}A^{ab}(\mathbf{v})\psi^*_a(\mathbf{v})u^{\mu}(\mathbf{v})\phi_b(\mathbf{v})\right] \tag{3.2}$$

and

$$(X^{\mu}\psi, \phi) = i\frac{M^2}{2}\int d\mu(\mathbf{v})\left[\frac{1}{M}A^{ab}(\mathbf{v})A^{cd}(\mathbf{v})\frac{\delta\psi^*_a(\mathbf{v})}{\delta v^d}u^{\mu}_{,c}(\mathbf{v})\phi_b(\mathbf{v})\right.$$
$$\left. + \frac{3}{2M}A^{ab}(\mathbf{v})\psi^*_a(\mathbf{v})u^{\mu}(\mathbf{v})\phi_b(\mathbf{v})\right] \tag{3.3}$$

or symbolically,

$$X^{\mu} = -\frac{i}{M}\left[A^{cd}(\mathbf{v})u^{\mu}_{,c}(\mathbf{v})\frac{\delta}{\delta v^d} + \frac{3}{2}u^{\mu}(\mathbf{v})\right] \tag{3.4}$$

This operator is clearly covariant (see Appendix C).

The momentum operator P^μ is defined as before in (2.10) and

$$[X^\mu, P^\nu] = -iA^{cd}(v)u^\mu_{,c}(\mathbf{v})u^\nu_{,d}(\mathbf{v}) = i[g^{\mu\nu} - u^\mu(\mathbf{v})u^\nu(\mathbf{v})] \tag{3.5}$$

or

$$[X^\mu, P^\nu] = i\left(g^{\mu\nu} - \frac{P^\mu P^\nu}{M^2}\right) \tag{3.6}$$

The orbital angular momentum is defined as before in (2.13) using the new X^μ, and the explicit expression for $L^{\mu\nu}$ is just that given in (2.16).

4. X^μ FOR GENERAL SPIN AND EITHER A DISCRETE OR A CONTINUOUS MASS SPECTRUM

The results of Sections 2 and 3 will now be generalized to include the case of a particle with arbitrary integral or half odd integral spin. For the most part, the mass variable M will be suppressed. For the case of integral spin, the generalization is straightforward. Simply use wave functions $\psi_{abc\cdots}(\mathbf{v})$ which are mass hyperboloid tensors of higher order and decompose them into irreducible components using the tensors $A^{ab}(\mathbf{v})$, $A_{ab}(\mathbf{v})$, $\epsilon^{abc}(\mathbf{v})$, and $\epsilon_{abc}(\mathbf{v})$. For example, for a second-order tensor $\psi_{ab}(\mathbf{v})$, one has

$$\psi_{ab}(\mathbf{v}) = \psi^{(0)}(\mathbf{v})A_{ab}(\mathbf{v}) + A^{cd}(\mathbf{v})\psi^{(1)}_c(\mathbf{v})\epsilon_{dab}(\mathbf{v}) + \psi^{(2)}_{ab}(\mathbf{v}) \tag{4.1}$$

where

$$\begin{aligned}
\psi^{(0)}(\mathbf{v}) &= \tfrac{1}{3}A^{ab}(\mathbf{v})\psi_{ab}(\mathbf{v}) \\
\psi^{(1)}_a(\mathbf{v}) &= \tfrac{1}{2}\epsilon_{abc}(\mathbf{v})A^{bd}(\mathbf{v})A^{ce}(\mathbf{v})\psi_{de}(\mathbf{v}) \\
\psi^{(2)}_{ab}(\mathbf{v}) &= \tfrac{1}{2}[\psi_{ab}(\mathbf{v}) + \psi_{ba}(\mathbf{v})] - \tfrac{1}{3}A_{ab}(\mathbf{v})A^{cd}(\mathbf{v})\psi_{cd}(\mathbf{v})
\end{aligned} \tag{4.2}$$

Then $\psi^{(2)}_{ab}(\mathbf{v})$ is a suitable wave function for a spin 2 particle. Moreover, the position-time operators are defined by (2.7) and (3.4), respectively, and the demonstrations of covariance and Hermiticity again go through with only minor modifications.

The presence of the metric tensor $A^{ab}(\mathbf{v})$ in the inner product (2.1) or (3.1) complicates the introduction of Pauli spinors required for the description of half odd integral spin. This difficulty can be overcome by employing at each point of the mass hyperboloid a basis normalized to unity, that is,

$$n_a{}^\mu(\mathbf{v}) = \left(1 - \frac{\mathbf{v}\cdot\mathbf{v}}{4}\right)u^\mu_{,a}(\mathbf{v}) \tag{4.3}$$

where

$$g_{\mu\nu}n_a{}^\mu(\mathbf{v})n_b{}^\nu(\mathbf{v}) = \delta_{ab} \tag{4.4}$$

The new wave functions for spin 1 $\psi^{(w)}_a(\mathbf{v})$, called Wigner wave functions, are related to the tensor wave functions by

$$\psi^{(w)}_a(\mathbf{v}) = \left(1 - \frac{\mathbf{v}\cdot\mathbf{v}}{4}\right)\psi_a(\mathbf{v}) \tag{4.5}$$

and the inner product becomes

$$(\psi, \phi) = \int d\mu(\mathbf{v})\psi_a^{(w)*}(\mathbf{v})\phi_a^{(w)}(\mathbf{v}) \tag{4.6}$$

where the repeated index is summed and there is no longer any distinction between upper and lower indices.

The representation of the Lorentz group is given by

$$[U(\Lambda)\psi]_a^{(w)}(\mathbf{v}) = [B^{-1}(\mathbf{v})\Lambda B(\Lambda^{-1}\mathbf{v})]_{ab}\psi_b^{(w)}(\Lambda^{-1}\mathbf{v}) \tag{4.7}$$

where $B(\mathbf{v})$ denotes the pure Lorentz transformation which takes $u^\mu(\mathbf{0})$ into $u^\mu(\mathbf{v})$ and

$$[B^{-1}(\mathbf{v})\Lambda B(\Lambda^{-1}\mathbf{v})]_{ab} = -n_a{}^\mu(\mathbf{v})\Lambda_{\mu\nu}n_b{}^\nu(\Lambda^{-1}\mathbf{v}) \tag{4.8}$$

is the well-known Wigner rotation (Moussa and Stora, 1968) [compare with (A.41)]. Clearly, the transformations (4.7) are unitary with respect to the inner product (4.6). Since

$$\begin{aligned}
&[U(\Lambda_1)\{U(\Lambda_2)\psi\}]_a^{(w)}(\mathbf{v}) \\
&\quad = [B^{-1}(\mathbf{v})\Lambda_1 B(\Lambda_1^{-1}\mathbf{v})]_{ab}[U(\Lambda_2)\psi]_b^{(w)}(\Lambda_1^{-1}\mathbf{v}) \\
&\quad = [B^{-1}(\mathbf{v})\Lambda_1 B(\Lambda_1^{-1}\mathbf{v})]_{ab}[B^{-1}(\Lambda_1^{-1}\mathbf{v})\Lambda_2 B(\Lambda_2^{-1}\Lambda^{-1}\mathbf{v})]_{bc}\psi_c^{(w)}(\Lambda_2^{-1}\Lambda_1^{-1}\mathbf{v}) \\
&\quad = [B^{-1}(\mathbf{v})(\Lambda_1\Lambda_2)B((\Lambda_1\Lambda_2)^{-1}\mathbf{v})]_{ab}\psi_b^{(w)}[(\Lambda_1\Lambda_2)^{-1}\mathbf{v}] \\
&\quad = [U(\Lambda_1\Lambda_2)\psi]_a^{(w)}(\mathbf{v})
\end{aligned} \tag{4.9}$$

the group property is also satisfied.

Because dv^a transforms as a tensor with an upper index, the combination

$$\frac{dv^a}{[1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})]} \tag{4.10}$$

transforms according to a Wigner rotation.

In order to define the covariant derivative for the Wigner wave functions, it is necessary to translate the equation for parallel transport of a tensor field along a specified curve into the new notation. Let $\psi_a(\mathbf{v} + d\mathbf{v}\| \to \mathbf{v})$ denote the components of the field ψ evaluated at $\mathbf{v} + d\mathbf{v}$ parallel translated to $\mathbf{v}$ along the infinitesimal curve defined by $d\mathbf{v}$. Then

$$\psi_a(\mathbf{v} + d\mathbf{v}\| \to \mathbf{v}) = \psi_a(\mathbf{v} + d\mathbf{v}) - \begin{Bmatrix} c \\ a\ b \end{Bmatrix}\psi_c(\mathbf{v})\, dv^b \tag{4.11}$$

Using (4.5) and (A.33) and retaining only terms up to first order in dv^a, one obtains

$$\psi_a^{(w)}(\mathbf{v} + d\mathbf{v}\| \to \mathbf{v}) = \psi_a^{(w)}(\mathbf{v} + d\mathbf{v}) - \frac{1}{2}\left\{\frac{v^a\, dv^b - dv^a v^b}{[1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})]}\right\}\psi_b^{(w)}(\mathbf{v}) \tag{4.12}$$

In Appendix D it is shown that (4.12) may be written in the form

$$\psi_a^{(w)}(\mathbf{v} + d\mathbf{v}\| \to \mathbf{v}) = R_{ab}(\mathbf{v}, \mathbf{v} + d\mathbf{v})\psi_b^{(w)}(\mathbf{v} + d\mathbf{v}) \tag{4.13}$$

where $R_{ab}(\mathbf{v}, \mathbf{v} + d\mathbf{v})$ is the Wigner rotation corresponding to the pure Lorentz transformation which takes $\mathbf{v} + d\mathbf{v}$ into $\mathbf{v}$.

The covariant derivative of the Wigner wave function is defined by

$$\frac{\delta}{\delta v^b}\psi_a^{(w)}(\mathbf{v}) = \lim_{\Delta v^b \to 0}\left\{\frac{\psi_a^{(w)}(\mathbf{v} + \Delta\mathbf{v}\| \to \mathbf{v}) - \psi_a^{(w)}(\mathbf{v})}{\Delta v^b/[1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})]}\right\} \tag{4.14}$$

Using (4.13) and (D.14), one obtains

$$\frac{\delta\psi_a^{(w)}(\mathbf{v})}{\delta v^b} = \left(1 - \frac{\mathbf{v}\cdot\mathbf{v}}{4}\right)\frac{\partial\psi_a^{(w)}(\mathbf{v})}{\partial v^b} - \frac{i}{2}v^c(S_{cb})_{ad}\psi_d^{(w)}(\mathbf{v}) \tag{4.15}$$

The operators X^μ and P^μ are given by

$$X^\mu = i\left[u^\mu(\mathbf{v})\frac{\partial}{\partial M} - \frac{1}{M}n_a{}^\mu(\mathbf{v})\left(\frac{\delta}{\delta v^a}\right)_w\right] \tag{4.16}$$

or

$$X^\mu = -\frac{i}{M}\left[n_a{}^\mu(\mathbf{v})\left(\frac{\delta}{\delta v^a}\right)_w + \frac{3}{2}u^\mu(\mathbf{v})\right] \tag{4.17}$$

and

$$P^\mu = Mu^\mu(\mathbf{v}) \tag{4.18}$$

where (4.16) and (4.17) correspond to (2.7) and (3.4), respectively. Again, the respective commutation relations (2.12) and (3.6) obtain.

These results may be readily generalized to the case of arbitrary integral or half odd integral spin s. A particle of spin s has a Wigner wave function $\psi_\lambda^{(w)}(\mathbf{v})$ with $2s + 1$ components, $\lambda \in \{-s, -s + 1, \ldots, s - 1, s\}$, which is square integrable with respect to the inner product

$$(\psi, \phi) = \int d\mu(\mathbf{v})\psi_\lambda^{(w)*}(\mathbf{v})\phi_\lambda^{(w)}(\mathbf{v}) \tag{4.19}$$

Under the Lorentz transformation Λ, these wave functions transform according to

$$[U(\Lambda)\psi]_\lambda^{(w)}(\mathbf{v}) = D_{\lambda\mu}^{(s)}[B^{-1}(\mathbf{v})\Lambda B(\Lambda^{-1}\mathbf{v})]\psi_\mu^{(w)}(\Lambda^{-1}\mathbf{v}) \tag{4.20}$$

where $D^{(s)}$ is the usual $(2s + 1) \times (2s + 1)$ unitary irreducible representation of the rotation group. Let S_{ab} denote the generators of $D^{(s)}$, then

$$D^{(s)}(R) = \exp\left[\frac{i}{2}\omega_{ab}S_{ab}\right] \tag{4.21}$$

where

$$R_{cd} = \exp\left[\frac{i}{2}\,\omega_{ab}(S_{ab})_{cd}\right] \tag{4.22}$$

The covariant derivative is defined by

$$\frac{\delta}{\delta v^a}\,\psi_\lambda^{(w)}(\mathbf{v}) = \lim_{\Delta v^a \to 0}\left\{\frac{D^{(s)}_{\lambda\mu}[R(\mathbf{v},\,\mathbf{v}+\Delta\mathbf{v})]\psi_\mu^{(w)}(\mathbf{v}+\Delta\mathbf{v}) - \psi_\lambda^{(w)}(\mathbf{v})}{\Delta v^a/[1 - \tfrac{1}{4}(\mathbf{v}\cdot\mathbf{v})]}\right\} \tag{4.23}$$

The explicit expression is

$$\frac{\delta\psi_\lambda^{(w)}(\mathbf{v})}{\delta v^a} = \left(1 - \frac{\mathbf{v}\cdot\mathbf{v}}{4}\right)\frac{\partial\psi_\lambda^{(w)}(\mathbf{v})}{\partial v^a} - \frac{i}{2}\,v^c(S_{ca})_{\lambda\mu}\psi_\mu^{(w)}(\mathbf{v}) \tag{4.24}$$

The expressions for the position-time operator X^μ are again given by (4.16) and (4.17) for the continuous and discrete mass spectrum cases, respectively. The orbital angular momentum operator $L^{\mu\nu}$ is given by

$$L^{\mu\nu} = i[u^\mu(\mathbf{v})n_a{}^\nu(\mathbf{v}) - u^\nu(\mathbf{v})n_a{}^\mu(\mathbf{v})]\left(\frac{\delta}{\delta v^a}\right)_w \tag{4.25}$$

in either case.

Clearly, (4.23) and (4.24) generalize in a straightforward way to cover the case of a tensor with an arbitrary number of spin indices of whatever spin types desired.

5. TOTAL ANGULAR MOMENTUM $J^{\mu\nu}$

The total angular momentum operator $J^{\mu\nu}$ is defined as generator of the representation of the Lorentz group. Expressions for both the special case of spin 1 (2.2) and the case of general spin s (4.20) will be given. Set

$$U(\Lambda) = \exp\left[\frac{i}{2}\,\omega_{\mu\nu}J^{\mu\nu}\right] \tag{5.1}$$

where

$$\Lambda^\mu{}_\nu = \exp\left[\frac{i}{2}\,\omega_{\rho\sigma}(I^{\rho\sigma})^\mu{}_\nu\right] \tag{5.2}$$

and

$$(I^{\rho\sigma})^{\mu\nu} = -i(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\nu}g^{\sigma\mu}) \tag{5.3}$$

The matrices $I^{\mu\nu}$ satisfy the commutation relations

$$[I^{\mu\nu}, I^{\rho\sigma}] = i(g^{\rho\mu}I^{\sigma\nu} + g^{\sigma\nu}I^{\rho\mu} - g^{\rho\nu}I^{\sigma\mu} - g^{\sigma\mu}I^{\rho\nu}) \tag{5.4}$$

The evaluation of the operator $J^{\mu\nu}$ is straightforward and is carried out in Appendix E first for the case of spin 1 and then for the case of general spin s. The result may be written

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu} \tag{5.5}$$

where $L^{\mu\nu} = X^\mu P^\nu - X^\nu P^\mu$, X^μ is the appropriate position-time operator, and $S^{\mu\nu}$ is just the projection of the spin matrices $(S_{ab})_{\lambda\mu}$ onto the basis $n_a{}^\rho(\mathbf{v})$. For the case of general spin s

$$S^{\mu\nu} = \tfrac{1}{2}[n_a{}^\mu(\mathbf{v})n_b{}^\nu(\mathbf{v}) - n_a{}^\nu(\mathbf{v})n_b{}^\mu(\mathbf{v})]S_{ab} \tag{5.6}$$

For the case of spin 1, $S^{\mu\nu}$ is given by (2.15).

Thus for a particle with given spin s, it is possible to define a position-time operator X^μ which is Hermitian and covariant, has the proper commutation relations with the momentum-energy operator P^μ, and leads in the standard way to an Hermitian, covariant orbital angular momentum operator $L^{\mu\nu}$, which in turn is suitably related to the total angular momentum operator $J^{\mu\nu}$. However, $[X^\mu, X^\nu] \neq 0$! Nevertheless, the operator X^μ defined above is the best possible given the restriction to a single spin s as the following general argument shows.

Let $J^{\mu\nu}$, P^μ, and X^μ be Hermitian operators which satisfy

$$\begin{aligned}
[J^{\mu\nu}, J^{\rho\sigma}] &= i(g^{\mu\rho}J^{\nu\sigma} + g^{\nu\sigma}J^{\mu\rho} - g^{\mu\sigma}J^{\nu\rho} - g^{\nu\rho}J^{\mu\sigma}) \\
[P^\lambda, J^{\mu\nu}] &= i(g^{\lambda\nu}P^\mu - g^{\lambda\mu}P^\nu) \\
[X^\lambda, J^{\mu\nu}] &= i(g^{\lambda\nu}X^\mu - g^{\lambda\mu}X^\nu) \\
[P^\mu, P^\nu] &= 0 \\
[X^\mu, X^\nu] &= 0
\end{aligned} \tag{5.7}$$

as well as *one* of the relations

$$\begin{aligned}
[X^\mu, P^\nu] &= ig^{\mu\nu} \\
[X^\mu, P^\nu] &= i\left(g^{\mu\nu} - \frac{P^\mu P^\nu}{M^2}\right)
\end{aligned} \tag{5.8}$$

[NOTE: The operators $J^{\mu\nu}$, P^μ, X^μ previously defined satisfy all of the conditions (5.7) and (5.8) except for $[X^\mu, X^\nu] = 0$.]

Then define the operator $L^{\mu\nu}$ as in (2.13). $L^{\mu\nu}$ is Hermitian since both P^μ and X^μ are. Moreover,

$$\begin{aligned}
[L^{\mu\nu}, L^{\rho\sigma}] &= i(g^{\mu\rho}L^{\nu\sigma} + g^{\nu\sigma}L^{\mu\rho} - g^{\mu\sigma}L^{\nu\rho} - g^{\nu\rho}L^{\mu\sigma}) \\
[X^\lambda, L^{\mu\nu}] &= i(g^{\lambda\nu}X^\mu - g^{\lambda\mu}X^\nu) \\
[P^\lambda, L^{\mu\nu}] &= i(g^{\lambda\nu}P^\mu - g^{\lambda\mu}P^\nu)
\end{aligned} \tag{5.9}$$

The relations (5.9) follow directly from the commutation relations of X^μ and P^μ. Next, define $S^{\mu\nu}$ by

$$S^{\mu\nu} = J^{\mu\nu} - L^{\mu\nu} \tag{5.10}$$

which is Hermitian since $J^{\mu\nu}$ and $L^{\mu\nu}$ are. From (5.7) and (5.9), one obtains

$$\begin{aligned} [X^\lambda, S^{\mu\nu}] &= 0 \\ [P^\lambda, S^{\mu\nu}] &= 0 \\ [L^{\rho\sigma}, S^{\mu\nu}] &= 0 \end{aligned} \tag{5.11}$$

Since

$$[J^{\mu\nu}, J^{\rho\sigma}] = [L^{\mu\nu}, L^{\rho\sigma}] + [S^{\mu\nu}, S^{\rho\sigma}] \tag{5.12}$$

it follows that

$$[S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\mu\rho}S^{\nu\sigma} + g^{\nu\sigma}S^{\mu\rho} - g^{\mu\sigma}S^{\nu\rho} - g^{\nu\rho}S^{\mu\sigma}) \tag{5.13}$$

Since the $S^{\mu\nu}$ are Hermitian and satisfy (5.13), they are the generators of a unitary representation of the homogeneous Lorentz group (Naimark, 1964).

The Pauli–Lubanski operator W_μ is given by

$$W_\mu \equiv -\tfrac{1}{2}e_{\mu\rho\sigma\tau}J^{\rho\sigma}P^\tau = -\tfrac{1}{2}e_{\mu\rho\sigma\tau}S^{\rho\sigma}P^\tau \tag{5.14}$$

If P^μ and $g^{\mu\nu}W_\mu W_\nu$ are simultaneously diagonalized, then

$$g^{\mu\nu}W_\mu W_\nu = -M^2 s(s+1) \tag{5.15}$$

where s is the particle spin quantum number. Clearly the operators X^μ, $L^{\mu\nu}$, and $S^{\mu\nu}$ do not commute with $g^{\mu\nu}W_\mu W_\nu$; consequently, in a basis in which the latter operator is diagonal, the former operators must have nonzero, off-diagonal matrix elements which connect states with different spin. Clearly, the operators X^μ, $L^{\mu\nu}$, and $S^{\mu\nu}$ defined earlier in this paper [(4.16), (4.17), (4.25), and (5.6)] are the restrictions of more general operators to a subspace with definite spin s, and this truncation explains why the restricted operators do not and cannot satisfy all of the desired commutation relations.

The principal conclusion to be drawn from the above discussion is that quantum mechanics is incompatible with *finite*-dimensional spinor fields given special relativity and the desirability of fully adequate space–time operators.

6. FOUR-DIMENSIONAL DENSITIES AND THE STATIC SPACE–TIME DESCRIPTION

There are two modes for describing events in space–time. The first views space–time as a single block and presents the description in terms of densities in four-dimensional space–time. The second considers a family of spacelike hypersurfaces or time slices and uses flux densities to describe conditions on

each hypersurface. The former viewpoint is developed in this section and the latter viewpoint is developed in the following section. Later, in Section 8, it will be argued that the customary kinematic formalism used in relativistic quantum mechanics is irrational because it is an *ad hoc* blend of these two formalisms which are mutually incompatible.

With the exception of the case of spin zero, it was shown in Section 5 that wave functions which transform according to a unitary representation of the homogeneous Lorentz group 0(1, 3) must be used if one requires the existence of a position-time operator X^μ which is fully adequate in the sense that the conditions (5.7) and (5.8) (especially, $[X^\mu, X^\nu] = 0$) are satisfied. Moreover, the wave functions must represent more than a single spin. Since a unitary representation may be decomposed into a direct sum/integral of unitary irreducible representations, it is natural to use a unitary irreducible representation of 0(1, 3). Since all such representations are infinite dimensional, the wave function represents an infinite spin tower of particles.

The unitary irreducible representations of 0(1, 3) (Naimark, 1964) may be labeled by a pair of numbers $[k_0, c]$. For a representation in the principal series

$$\begin{array}{ll} c = i\rho & -\infty < \rho < \infty \\ k_0 \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots\} & \end{array} \tag{6.1}$$

For a representation in the complementary series

$$0 < c < 1 \qquad k_0 = 0 \tag{6.2}$$

The identity representation corresponds to the pair [0, 1]. A basis for a representation in either the principal or complementary series may be labeled by a pair of indices (j, m) which range over the values

$$\begin{array}{ll} m = -j & -j+1, \ldots, j-1, j \\ j = k_0 & k_0 + 1, k_0 + 2, \ldots \end{array} \tag{6.3}$$

and the representation is infinite dimensional. The values of the Casimir operators in the representation $[k_0, c]$ are given by

$$\begin{aligned} C_1 &= \tfrac{1}{2} g_{\mu\rho} g_{\nu\sigma} S^{\mu\nu} S^{\rho\sigma} = k_0^2 + c^2 - 1 \\ C_2 &= \tfrac{1}{4} e_{k\lambda\mu\nu} S^{k\lambda} S^{\mu\nu} = -ik_0 c \end{aligned} \tag{6.4}$$

The physical meaning of the quantum numbers (ρ, k_0) will not be discussed here. It would seem that the use of wave functions which transform under a representation of 0(1, 3), whether finite or infinite dimensional, implies that one's model of a particle is a *four*-dimensional structure moving along a world line in Minkowski space–time rather than a world line tube with a small three-dimensional cross-section. The reader is referred to the end of Section 10 for further discussion of this point. In any case, in this and

the following section, it is assumed that physical states correspond to rays in the Hilbert space of infinite-component, complex-valued wave functions defined on Minkowski space-time, $\psi_{jm}(x)$, which are square integrable with respect to the inner product

$$(\psi, \phi) = \sum_{jm} \int d^4x \psi^*_{jm}(x) \phi_{jm}(x) \tag{6.5}$$

which transform under the Lorentz group according to one of the above unitary irreducible representations

$$[U(\Lambda)\psi]_{jm}(x) = U(\Lambda)_{jm,j'm'} \psi_{j'm'}(\Lambda^{-1}x) \tag{6.6}$$

and for which the mass operator $(-\Box)$ is positive. Denote the momentum space wave functions for positive and negative frequency by $\psi^{(+)}_{s\lambda}(M, \mathbf{v})$ and $\psi^{(-)}_{s\lambda}(M, \mathbf{v})$, respectively. Under a Lorentz transformation, both of these wave functions transform according to (4.20). [The usual scheme employs the conjugate representation $D^{(s)*}$ for the negative frequency wave function. The reason for this modification will be made clear below.] Under a translation, $x \to x + a$, these wave functions transform according to

$$[U(a)\psi^{(\pm)}]_{s\lambda}(M, \mathbf{v}) = \exp[\pm i g_{\mu\nu} M u^\mu(\mathbf{v}) a^\nu] \psi^{(\pm)}_{s\lambda}(M, \mathbf{v}) \tag{6.7}$$

The wave function $\psi_{jm}(x)$ may be expressed in terms of the wave functions $\psi^{(\pm)}_{s\lambda}(M, \mathbf{v})$ through the expansion

$$\psi_{jm}(x) = \sum_{s\lambda} \int d\mu(M, \mathbf{v}) [f^{(+)}_{jm}(x; M, \mathbf{v}, s, \lambda) \psi^{(+)}_{s\lambda}(M, \mathbf{v}) + f^{(-)}_{jm}(x; M, \mathbf{v}, s, \lambda) \psi^{(-)}_{s\lambda}(M, \mathbf{v})] \tag{6.8}$$

where

$$f^{(\pm)}_{jm}(x; M, \mathbf{v}, s, \lambda) = \frac{1}{(2\pi)^2} \exp[\mp i g_{\mu\nu} M u^\mu(\mathbf{v}) x^\nu] B_{jm}(\mathbf{v}, s, \lambda) \tag{6.9}$$

and the functions $B_{jm}(\mathbf{v}, s, \lambda)$ are defined by (F.2) in Appendix F.

Using the orthonormality relations

$$\sum_{jm} \int d^4x f^{(\pm)*}_{jm}(x; M', \mathbf{v}', s', \lambda') f^{(\pm)}_{jm}(x; M, \mathbf{v}, s, \lambda) = \delta_{s's} \delta_{\lambda'\lambda} \delta(M' - M) \left[\frac{1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})}{M}\right]^3 \delta^3(\mathbf{v}' - \mathbf{v}) \tag{6.10}$$

$$\sum_{jm} \int d^4x f^{(\pm)*}_{jm}(x; M', \mathbf{v}', s', \lambda') f^{(\mp)}_{jm}(x; M, v, s, \lambda) = 0$$

one obtains for the relations inverse to the expansion (6.8)

$$\psi_{s\lambda}^{(\pm)}(M, \mathbf{v}) = \sum_{jm} \int d^4x f_{jm}^{(\pm)*}(x; M, \mathbf{v}, s, \lambda)\psi_{jm}(x) \tag{6.11}$$

Moreover,

$$\begin{aligned}(\psi, \phi) &= \sum_{jm} \int d^4x \psi_{jm}^*(x)\phi_{jm}(x) \\ &= \sum_{s\lambda} \int d\mu(M, \mathbf{v})[\psi_{s\lambda}^{(+)*}(M, \mathbf{v})\phi_{s\lambda}^{(+)}(M, \mathbf{v}) + \psi_{s\lambda}^{(-)*}(M, \mathbf{v})\phi_{s\lambda}^{(-)}(M, \mathbf{v})]\end{aligned} \tag{6.12}$$

The wave functions (6.9) satisfy the completeness condition

$$\begin{aligned}\sum_{s\lambda} \int d\mu(M, \mathbf{v})[f_{j'm'}^{(+)}(x'; M, \mathbf{v}, s, \lambda) f_{jm}^{(+)*}(x; M, \mathbf{v}, s, \lambda) \\ + f_{j'm'}^{(-)}(x'; M, \mathbf{v}, s, \lambda) f_{jm}^{(-)*}(x; M, \mathbf{v}, s, \lambda)] = \delta_{j'j}\delta_{m'm}\underline{\delta}(x' - x)\end{aligned} \tag{6.13}$$

where

$$\underline{\delta}(x) = \frac{1}{(2\pi)^4} \int d\mu(M, \mathbf{v})\{\exp[-ig_{\mu\nu}Mu^\mu(\mathbf{v})x^\nu] + \exp[ig_{\mu\nu}Mu^\mu(v)x^\nu]\} \tag{6.14}$$

The interpretation of $\underline{\delta}(x)$ is important. It is *not* a propagator. It is just the restriction of the four-dimensional Dirac δ function $\delta^4(x)$ to the case in which only timelike momenta are included. For the class of wave functions with a timelike momentum spectrum, $\underline{\delta}(x)$ plays the role of $\delta^4(x)$. In particular, $\underline{\delta}(x)$ satisfies the relations

$$\begin{aligned}\underline{\delta}(-x) &= \underline{\delta}(x) \\ \psi_{jm}(x) &= \int d^4x' \underline{\delta}(x - x')\psi_{jm}(x') \\ \underline{\delta}(x - x') &= \int d^4x'' \underline{\delta}(x - x'')\underline{\delta}(x'' - x')\end{aligned} \tag{6.15}$$

Variational derivatives of functionals of the fields may be defined by setting

$$\frac{\delta\psi_{jm}(x)}{\delta\psi_{j'm'}(x')} = \frac{\delta\psi_{jm}^*(x)}{\delta\psi_{j'm'}^*(x')} = \delta_{j'j}\delta_{m'm}\underline{\delta}(x' - x) \tag{6.16}$$

$$\frac{\delta\psi_{s\lambda}^{(\pm)}(M, \mathbf{v})}{\delta\psi_{s'\lambda'}^{(\pm)}(M', \mathbf{v}')} = \frac{\delta\psi_{s\lambda}^{(\pm)*}(M, \mathbf{v})}{\delta\psi_{s'\lambda'}^{(\pm)*}(M', \mathbf{v}')} = \delta_{s's}\delta_{\lambda'\lambda}\delta(M' - M)\left[\frac{1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})}{M}\right]^3 \delta^3(\mathbf{v}' - \mathbf{v})$$

and by assigning zero to the other possibilities. The anti-Poisson bracket (+) and the Poisson bracket (−) of two functionals f and g of the fields $\psi_{jm}(x)$ and $\psi^*_{jm}(x)$ are defined by

$$\{f, g\}_\pm = \sum_{jm} \int d^4x \left[\frac{\delta f}{\delta \psi_{jm}(x)} \frac{\delta g}{\delta \psi^*_{jm}(x)} \pm \frac{\delta g}{\delta \psi_{jm}(x)} \frac{\delta f}{\delta \psi^*_{jm}(x)} \right] \tag{6.17}$$

For the basic fields, one may use either type of bracket. One obtains

$$\{\psi_{j'm'}(x'), \psi^*_{jm}(x)\}_\pm = \delta_{j'j}\delta_{m'm}\delta(x' - x) \tag{6.18}$$

while the other brackets are zero. Whenever at least one of the two functionals f and g is of even degree in the basic fields, the Poisson bracket (−) should be employed.

It follows from (6.11) that

$$\{\psi^{(+)}_{s'\lambda'}(M', \mathbf{v}'), \psi^{(+)*}_{s\lambda}(M, \mathbf{v})\}_\pm = \delta_{s's}\delta_{\lambda'\lambda}\delta(M' - M)\left[\frac{1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})}{M}\right]^3 \delta^3(\mathbf{v}' - \mathbf{v}) \tag{6.19}$$

$$\{\psi^{(-)}_{s'\lambda'}(M', \mathbf{v}'), \psi^{(-)*}_{s\lambda}(M, \mathbf{v})\}_\pm = \delta_{s's}\delta_{\lambda'\lambda}\delta(M' - M)\left[\frac{1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})}{M}\right]^3 \delta^3(\mathbf{v}' - \mathbf{v})$$

while the other Poisson brackets are zero. In terms of the momentum wave functions, the Poisson bracket (6.17) is given by

$$\begin{aligned}\{f, g\}_\pm = \sum_{s\lambda} \int d\mu(M, \mathbf{v}) \\ \times \left[\left(\frac{\delta f}{\delta \psi^{(+)}_{s\lambda}(M, \mathbf{v})} \frac{\delta g}{\delta \psi^{(+)*}_{s\lambda}(M, \mathbf{v})} + \frac{\delta f}{\delta \psi^{(+)}_{s\lambda}(M, \mathbf{v})} \frac{\delta g}{\delta \psi^{(+)*}_{s\lambda}(M, \mathbf{v})} \right) \right. \\ \left. \pm \left(\frac{\delta g}{\delta \psi^{(-)}_{s\lambda}(M, \mathbf{v})} \frac{\delta f}{\delta \psi^{(-)*}_{s\lambda}(M, \mathbf{v})} + \frac{\delta g}{\delta \psi^{(-)}_{s\lambda}(M, \mathbf{v})} \frac{\delta f}{\delta \psi^{(-)*}_{s\lambda}(M, \mathbf{v})} \right) \right]\end{aligned} \tag{6.20}$$

The operators X^μ, P^μ, $L^{\mu\nu}$, and $J^{\mu\nu}$ are most conveniently described with respect to the wave functions $\psi^{(\pm)}_{jm}(M, \mathbf{v})$:

$$\psi^{(\pm)}_{jm}(M, \mathbf{v}) = B_{jm}(\mathbf{v}, s, \lambda)\psi^{(\pm)}_{s\lambda}(M, \mathbf{v}) \tag{6.21}$$

where the $B_{jm}(\mathbf{v}, s, \lambda)$ are given by (F.2). Under a Lorentz transformation, these wave functions transform according to

$$[U(\Lambda)\psi^{(\pm)}]_{jm}(M, \mathbf{v}) = U_{jm;j'm'}(\Lambda)\psi^{(\pm)}_{j'm'}(M, \Lambda^{-1}\mathbf{v}) \tag{6.22}$$

In this basis, the position-time operator is simply the momentum-energy gradient (A.38):

$$X^\mu \psi^{(\pm)}_{jm}(M, \mathbf{v}) = \pm i \left[u^\mu(\mathbf{v}) \frac{\partial}{\partial M} - \frac{1}{M} A^{ab}(\mathbf{v}) u^\mu_{,a}(\mathbf{v}) \frac{\partial}{\partial v^b} \right] \psi^{(\pm)}_{jm}(M, \mathbf{v}) \tag{6.23}$$

In agreement with (6.7), the operator P^μ is given by

$$P^\mu \psi^{(\pm)}_{jm}(M, \mathbf{v}) = \pm M u^\mu(\mathbf{v}) \psi^{(\pm)}_{jm}(M, \mathbf{v}) \tag{6.24}$$

The operators for orbital angular momentum $L^{\mu\nu}$ and total angular momentum $J^{\mu\nu}$ are given by

$$\begin{aligned} L^{\mu\nu}\psi^{(\pm)}_{jm}(M, \mathbf{v}) &= (X^\mu P^\nu - X^\nu P^\mu)\psi^{(\pm)}_{jm}(M, \mathbf{v}) \\ &= iA^{ab}(\mathbf{v})[u^\mu(\mathbf{v})u^\nu_{,a}(\mathbf{v}) - u^\nu(\mathbf{v})u^\mu_{,a}(\mathbf{v})] \frac{\partial}{\partial v^b} \psi^{(\pm)}_{jm}(M, \mathbf{v}) \end{aligned} \tag{6.25}$$

and

$$J^{\mu\nu}\psi^{(\pm)}_{jm}(M, \mathbf{v}) = L^{\mu\nu}\psi^{(\pm)}_{jm}(M, \mathbf{v}) + S^{\mu\nu}_{jm;j'm'}\psi^{(\pm)}_{j'm'}(M, \mathbf{v}) \tag{6.26}$$

The operators X^μ, P^μ, $L^{\mu\nu}$, $J^{\mu\nu}$, and $S^{\mu\nu}$ defined above are Hermitian and satisfy the relations given in (5.7)–(5.9), (5.11), and (5.13). These operators are related to those constructed in Sections 2–4 in the following way. Transform the expressions (6.23)–(6.26) to the basis $\psi^{(\pm)}_{s\lambda}(M, \mathbf{v})$. Then discard the terms which are not diagonal in the spin s. The operators so obtained are just those defined in Sections 2–4. The reason for the peculiar commutation relations of these truncated operators is now clear.

To each operator $\mathrm{Op} \in \{X^\mu, P^\mu, L^{\mu\nu}, J^{\mu\nu}, S^{\mu\nu}\}$ there corresponds a functional $\mathrm{Op}\,[\psi^*, \psi]$ given by[2]

$$\begin{aligned} \mathrm{Op}\,[\psi^*, \psi] = \sum_{jm} \int d\mu(M, \mathbf{v})[&\psi^{(+)*}_{jm}(M, \mathbf{v})\, \mathrm{Op}\, \psi^{(+)}_{jm}(M, \mathbf{v}) \\ &+ \psi^{(-)*}_{jm}(M, \mathbf{v})\, \mathrm{Op}\, \psi^{(-)}_{jm}(M, \mathbf{v})] \end{aligned} \tag{6.27}$$

For example,

$$\begin{aligned} P^\mu[\psi^*, \psi] = \sum_{s\lambda} \int d\mu(M, \mathbf{v}) M u^\mu(\mathbf{v})[&\psi^{(+)*}_{s\lambda}(M, \mathbf{v})\psi^{(+)}_{s\lambda}(M, \mathbf{v}) \\ &- \psi^{(-)*}_{s\lambda}(M, \mathbf{v})\psi^{(-)}_{s\lambda}(M, \mathbf{v})] \end{aligned} \tag{6.28}$$

One can readily show that

$$\{P^\mu[\psi^*, \psi], \psi_{jm}(x)\}_- = -i\partial^\mu \psi_{jm}(x) \tag{6.29}$$

and

$$\{P^\mu[\psi^*, \psi], P^\nu[\psi^*, \psi]\}_- = 0 \tag{6.30}$$

An entire set of such relations exists which corresponds to the expressions (6.23)–(6.26) and to the commutation relations (5.7)–(5.11) and (5.13).

[2] A corresponding "matrix element" $\mathrm{Op}\,[\psi^*, \phi]$ may also be defined.

Using (6.8) and (6.21), the operators given by (6.23)–(6.26) may be reexpressed in terms of the basis $\psi_{jm}(x)$. Then, the functionals (6.27) have the form of integrals over space–time densities

$$\text{Op}\,[\psi^*, \psi] = \sum_{jm} \int d^4x \psi^*_{jm}(x)\,\text{Op}\,\psi_{jm}(x) \tag{6.31}$$

Neither the operator P^0 nor the corresponding functional $P^0[\psi^*, \psi]$ is positive definite. This is reasonable since it is only the functional for the energy *flux* through a spacelike hypersurface which should be positive definite. Functionals related to flux densities are discussed in Section 7.

The eigenfunctions of the position-time operator in the basis (6.21) are

$$\psi^{(\pm)}_{jm}(M, \mathbf{v}; x, \gamma) = \alpha_{jm}(\gamma) \exp\,[\mp i g_{\mu\nu} M u^\mu(\mathbf{v}) x^\nu] \tag{6.32}$$

where the complex numbers $\alpha_{jm}(\gamma)$ are the components of a vector in the infinite-dimensional 0(1, 3) group representation space and where γ denotes the eigenvalues of two independent, noninvariant, commuting operators formed from the $S^{\mu\nu}$. (The eigenvalues of the Casimir operators are already fixed by the choice of representation.) The inner product (6.12) gives

$$\begin{aligned}(\psi(x', \gamma'), \psi(x, \gamma)) = \sum_{jm} \int d\mu(M, \mathbf{v}) [&\psi^{(+)*}_{jm}(M, \mathbf{v}; x', \gamma') \psi^{(+)}_{jm}(M, \mathbf{v}; x, \gamma) \\ &+ \psi^{(-)*}_{jm}(M, \mathbf{v}; x', \gamma') \psi^{(-)}_{jm}(M, \mathbf{v}; x, \gamma)] \\ = \delta_{\gamma'\gamma} \underline{\delta}(x' - x)\end{aligned} \tag{6.33}$$

Thus, wave functions with different space–time eigenvalues are not orthogonal. However, since the role of δ^4 is replaced by $\underline{\delta}$ for the space of functions with timelike momentum spectrum [see (6.15)], these wave functions are as orthogonal as possible given the exclusion of lightlike and spacelike momenta.

From a mathematical point of view, the above discussion of the Fourier transform on Minkowski space–time may be extended to include functions with a lightlike and spacelike momentum spectrum (see Appendix G). If this is done, then complete sums over *all* momenta will yield the four-dimensional δ function instead of the distribution $\underline{\delta}$, and eigenfunctions of the position-time operator which are properly orthogonal can be defined. This formalism very closely parallels that used in Euclidean space. In particular, the definition of the position-time operator given in this section is the direct analog of the definition of the position operator usually employed in the Euclidean case. Furthermore, the fact that either Poisson brackets or anti-Poisson brackets may be used for the basic fields [(6.17), (6.18)] indicates that there is no spin-statistics theorem. This absence of a spin-statistics relation also obtains in the Euclidean case. The construction of position operators in the Euclidean case is discussed in Section 9.

7. THE SPACELIKE HYPERSURFACE DESCRIPTION AND FLUX DENSITIES

In Section 6 the fundamental kinematic formalism for the description of particle states in space–time was developed. In that formalism, the events of space–time, past, present, and future, are described all at once as a block. In order to examine the development of a system in time, it is necessary to describe the same space–time events in terms of quantities defined on the members of a family of spacelike hypersurfaces which form a disjoint decomposition of space–time. The purpose of this section is to develop the kinematic formalism required for such a description. The quantities appropriate for the overall space–time description are four-dimensional space–time densities and the integrals of such densities over four-dimensional volume elements. For the description of time evolution, the relevant quantities are also four-dimensional densities and integrals of such densities. However, the basic quantities are now one-dimensional densities with respect to the mass spectrum and three-dimensional *flux* densities with respect to the hypersurfaces. The momentum space description is the same for both cases. It is emphasized that the formalism may be developed for *either* a continuous *or* a discrete mass spectrum. The formulas given are appropriate for the case of a continuous mass spectrum. For the most part, simple restriction of these formulas and a change of normalization yield the formulas appropriate to the case of a discrete mass. Cases where this is not so are noted.

The expansion (6.8) may be decomposed into an integral over fields with definite mass:

$$\psi_{jm}(x) = \int \frac{dM^2}{(2\pi)^{1/2}} \psi_{jm}^{(M^2)}(x) \tag{7.1}$$

where

$$\psi_{jm}^{(M^2)}(x) = \left(\frac{\pi}{2}\right)^{1/2} M^2 \sum_{s\lambda} \int d\mu(\mathbf{v})[f_{jm}^{(+)}(x; M, \mathbf{v}, s, \lambda)\psi_{s\lambda}^{(+)}(M, \mathbf{v}) + f_{jm}^{(-)}(x; M, \mathbf{v}, s, \lambda)\psi_{s\lambda}^{(+)}(M, \mathbf{v})] \tag{7.2}$$

and

$$(\Box + M^2)\psi_{jm}^{(M^2)}(x) = 0 \tag{7.3}$$

Given two wave functions $\psi_{jm}^{(M^2)}(x)$ and $\phi_{jm}^{(M^2)}(x)$ which satisfy (7.3), one may define the four-vector flux density

$$j^\mu(M^2; x) = i \sum_{jm} \psi_{jm}^{(M^2)*}(x) \overleftrightarrow{\partial_\mu} \phi_{jm}^{(M^2)}(x) \tag{7.4}$$

where

$$A \overleftrightarrow{\partial^\mu} B = A \frac{\partial B}{\partial x_\mu} - \frac{\partial A}{\partial x_\mu} B \tag{7.5}$$

for arbitrary fields A and B. The flux density (7.4) is conserved

$$\partial_\mu j^\mu(M^2; x) = 0 \tag{7.6}$$

Then one may define a sesquilinear form $\langle \psi^{(M^2)}, \phi^{(M^2)} \rangle$ by

$$\langle \psi^{(M^2)}, \phi^{(M^2)} \rangle = i \sum_{jm} \int_\sigma d\sigma_\mu(x) \psi^{(M^2)*}_{jm}(x) \overleftrightarrow{\partial^\mu} \phi^{(M^2)}_{jm}(x) \tag{7.7}$$

It follows from (7.6) that (7.7) is independent of the spacelike surface σ.

The wave functions (6.9) satisfy

$$((2\pi)^{1/2} f^{(\pm)}(M, \mathbf{v}', s', \lambda'), (2\pi)^{1/2} f^{(\pm)}(M, \mathbf{v}, s, \lambda)) = \pm \delta_{s's} \delta_{\lambda'\lambda} 2M \left[\frac{1 - \frac{1}{4}(\mathbf{v}' \cdot \mathbf{v})}{M} \right]^3 \delta^3(\mathbf{v}' - \mathbf{v}) \tag{7.8}$$

$$((2\pi)^{1/2} f^{(\pm)}(M, \mathbf{v}', s', \lambda'), (2\pi)^{1/2} f^{(\mp)}(M, \mathbf{v}, s, \lambda)) = 0$$

The normalization used in (7.8) may be related to the usual one by means of

$$\frac{d^3\mathbf{p}}{2\omega_p} = \frac{1}{2M} \left[\frac{M}{1 - \frac{1}{4}(\mathbf{v} \cdot \mathbf{v})} \right]^3 d^3\mathbf{v} \tag{7.9}$$

The inverse of (7.2) is

$$\psi^{(\pm)}_{s\lambda}(M, \mathbf{v}) = \pm ((2\pi)^{1/2} f^{(\pm)}(M, \mathbf{v}, s, \lambda), \psi^{(M^2)}) \tag{7.10}$$

Using (7.2), (7.7), and (7.8), one obtains

$$\langle \psi^{(M^2)}, \phi^{(M^2)} \rangle = \frac{M^2}{2} \sum_{s\lambda} \int d\mu(\mathbf{v}) [\psi^{(+)*}_{s\lambda}(M, \mathbf{v}) \phi^{(+)}_{s\lambda}(M, \mathbf{v}) - \psi^{(-)*}_{s\lambda}(M, \mathbf{v}) \phi^{(-)}_{s\lambda}(M, \mathbf{v})] \tag{7.11}$$

In contrast to (6.5), the sesquilinear form (7.7) [or (7.11)] is not positive definite; consequently, the positive and negative frequency wave functions enter the corresponding completeness condition with (+) and (−) signs, respectively. Thus the completeness condition corresponding to (7.8) is

$$\frac{M^2}{2} \sum_{s\lambda} \int d\mu(\mathbf{v}) [(2\pi)^{1/2} f^{(+)}_{j'm'}(x'; M, \mathbf{v}, s, \lambda)(2\pi)^{1/2} f^{(+)*}_{jm}(x; M, \mathbf{v}, s, \lambda) - (2\pi)^{1/2} f^{(-)}_{j'm'}(x'; M, \mathbf{v}, s, \lambda)(2\pi)^{1/2} f^{(-)*}_{jm}(x; M, \mathbf{v}, s, \lambda)] = \delta_{j'j} \delta_{m'm} i\Delta(x' - x; M^2) \tag{7.12}$$

where Δ is the usual *causal* propagator for a scalar field of mass M given by

$$\Delta(x; M^2) = \frac{-i}{(2\pi)^3} \int \frac{d^3\mathbf{p}}{2\omega_p} [\exp(-ip \cdot x) - \exp(ip \cdot x)] \tag{7.13}$$

where $\omega_p = (M^2 + \mathbf{p}^2)^{1/2}$ and the four momentum p is related to M and $\mathbf{v}$ by (A.7).

Denote the values of the field (7.2) and its normal derivative for the particular spacelike surface given by $x^0 = 0$ by

$$\psi_{jm}^{(M^2)}(\mathbf{x}) = \psi_{jm}^{(M^2)}(x)|_{x^0=0}$$
$$\dot{\psi}_{jm}^{(M^2)}(\mathbf{x}) = \frac{\partial \psi_{jm}^{(M^2)}(x)}{\partial x^0}\bigg|_{x^0=0} \tag{7.14}$$

Then

$$\psi_{jm}^{(M^2)}(x^0, \mathbf{x}) = -\int d^3\mathbf{x}' \Big[\Delta(x^0, \mathbf{x} - \mathbf{x}'; M^2)\dot{\psi}_{jm}^{(M^2)}(\mathbf{x}') + \frac{\partial \Delta(x^0, \mathbf{x} - \mathbf{x}'; M^2)}{\partial x^0}\psi_{jm}^{(M^2)}(\mathbf{x}')\Big] \tag{7.15}$$

which satisfies (7.3) and the initial conditions (7.14). The sesquilinear form (7.7) may also be expressed in terms of the fields (7.14):

$$\langle \psi^{(M^2)}, \phi^{(M^2)} \rangle = i \sum_{jm} \int_{x^0=0} d^3\mathbf{x}[\psi_{jm}^{(M^2)*}(\mathbf{x})\dot{\phi}_{jm}^{(M^2)}(\mathbf{x}) - \dot{\psi}_{jm}^{(M^2)*}(\mathbf{x})\phi_{jm}^{(M^2)}(\mathbf{x})] \tag{7.16}$$

The scalar product (6.5) may also be expressed in terms of the fields (7.14) by means of (6.12) and (7.10). The result is

$$\begin{aligned}(\psi, \phi) = \int dM^2 \int d^3\mathbf{x} d^3\mathbf{y} \sum_{jm} \\ \times \{\dot{\psi}_{jm}^{(M^2)*}(\mathbf{x})\Delta_1(\mathbf{x} - \mathbf{y}; M^2)\dot{\phi}_{jm}^{(M^2)}(\mathbf{y}) \\ + [\nabla_x \psi_{jm}^{(M^2)}(\mathbf{x})]^* \cdot \Delta_1(\mathbf{x} - \mathbf{y}; M^2)[\nabla_y \phi_{jm}^{(M^2)}(\mathbf{y})] \\ + M^2 \psi_{jm}^{(M^2)*}(\mathbf{x})\Delta_1(\mathbf{x} - \mathbf{y}; M^2)\phi_{jm}^{(M^2)}(\mathbf{y})\}\end{aligned} \tag{7.17}$$

where

$$\Delta_1(\mathbf{x} - \mathbf{y}; M^2) = \Delta_1(x - y; M^2)|_{x^0=y^0} \tag{7.18}$$

and

$$\Delta_1(x; M^2) = \frac{1}{(2\pi)^3}\int \frac{d^3\mathbf{p}}{2\omega_p}[\exp(-ip\cdot x) + \exp(ip\cdot x)] \tag{7.19}$$

Functionals of $\psi_{jm}(x)$ and $\psi_{jm}^*(x)$ may be expressed as functionals of $\psi_{jm}^{(M^2)}(\mathbf{x})$, $\dot{\psi}_{jm}^{(M^2)}(\mathbf{x})$ and $\psi_{jm}^{(M^2)*}(\mathbf{x})$, $\dot{\psi}_{jm}^{(M^2)*}(\mathbf{x})$ by means of (7.14) and (7.1). Then the appropriate functional derivatives are

$$\begin{aligned}\frac{\delta\psi_{jm}^{(M^2)}(\mathbf{x})}{\delta\psi_{j'm'}^{(M'^2)}(\mathbf{x}')} = \frac{\delta\psi_{jm}^{(M^2)*}(\mathbf{x})}{\delta\psi_{j'm'}^{(M'^2)*}(\mathbf{x}')} = \delta_{j'j}\delta_{m'm}\delta(M'^2 - M^2)\delta^3(\mathbf{x}' - \mathbf{x}) \\ \frac{\delta\dot{\psi}_{jm}^{(M^2)}(\mathbf{x})}{\delta\dot{\psi}_{j'm'}^{(M'^2)}(\mathbf{x}')} = \frac{\delta\dot{\psi}_{jm}^{(M^2)*}(\mathbf{x})}{\delta\dot{\psi}_{j'm'}^{(M'^2)*}(\mathbf{x}')} = \delta_{j'j}\delta_{m'm}\delta(M'^2 - M^2)\delta^3(\mathbf{x}' - \mathbf{x})\end{aligned} \tag{7.20}$$

while the other derivatives vanish. The variational derivatives for the momentum space wave functions remain the same as those given in (6.16).

Anti-Poisson brackets (+) and Poisson brackets (−) appropriate to the hypersurface description may be defined by

$$[f, g]_{PB\pm} = \int dM^2 \sum_{jm} \int d^3\mathbf{x} \times \left\{ \left[\frac{\delta f}{\delta \psi_{jm}^{(M^2)}(\mathbf{x})} \frac{\delta g}{\delta \dot{\psi}_{jm}^{(M^2)*}(\mathbf{x})} - \frac{\delta f}{\delta \dot{\psi}_{jm}^{(M^2)}(\mathbf{x})} \frac{\delta g}{\delta \psi_{jm}^{(M^2)*}(\mathbf{x})} \right] \pm \left[\frac{\delta g}{\delta \psi_{jm}^{(M^2)}(\mathbf{x})} \frac{\delta f}{\delta \dot{\psi}_{jm}^{(M^2)*}(\mathbf{x})} - \frac{\delta g}{\delta \dot{\psi}_{jm}^{(M^2)}(\mathbf{x})} \frac{\delta f}{\delta \psi_{jm}^{(M^2)*}(\mathbf{x})} \right] \right\} \tag{7.21}$$

Note that *each* of the two terms separated by $(\pm)$ is time independent. The brackets defined by (7.21) are quite distinct from those defined by (6.17).

Using (7.1), (7.15), and (7.20), one finds that the basic fields satisfy

$$[\psi_{j'm'}(x'), \psi_{jm}^*(x)]_{PB\pm} = \delta_{j'j}\delta_{m'm} \int \frac{dM^2}{2\pi} \Delta(x' - x; M^2) \tag{7.22}$$

where Δ is the *causal* propagator given by (7.13). Again, *either* type of bracket may be used for the basic fields, while the Poisson bracket $(-)$ should be used whenever at least one of the two functionals f and g is of even degree in the basic fields. There is no connection between spin and statistics inherent in the formalism.

The brackets of the momentum fields may be computed using (7.10). Those which are nonzero are

$$\begin{aligned} &i[\psi_{s'\lambda'}^{(\pm)}(M', \mathbf{v}'), \psi_{s\lambda}^{(\pm)*}(M, \mathbf{v})]_{PB+} \\ &\quad = \pm \delta_{s's}\delta_{\lambda'\lambda}\delta(M'^2 - M^2)2M\left[\frac{1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})}{M}\right]^3 \delta^3(\mathbf{v}' - \mathbf{v}) \\ &\quad = i[\psi_{s'\lambda'}^{(\pm)}(M', \mathbf{v}'), \psi_{s\lambda}^{(\pm)*}(M, \mathbf{v})]_{PB-} \end{aligned} \tag{7.23}$$

In terms of the momentum fields the brackets defined by (7.21) are given by

$$i[f, g]_{PB\pm} = \sum_{s\lambda} \int d\mu(M, \mathbf{v}) \times \left\{ \left[\frac{\delta f}{\delta \psi_{s\lambda}^{(+)}(M, \mathbf{v})} \frac{\delta g}{\delta \psi_{s\lambda}^{(+)*}(M, \mathbf{v})} - \frac{\delta f}{\delta \psi_{s\lambda}^{(-)}(M, \mathbf{v})} \frac{\delta g}{\delta \psi_{s\lambda}^{(-)*}(M, \mathbf{v})} \right] \pm \left[\frac{\delta g}{\delta \psi_{s\lambda}^{(+)}(M, \mathbf{v})} \frac{\delta f}{\delta \psi_{s\lambda}^{(+)*}(M, \mathbf{v})} - \frac{\delta g}{\delta \psi_{s\lambda}^{(-)}(M, \mathbf{v})} \frac{\delta f}{\delta \psi_{s\lambda}^{(-)*}(M, \mathbf{v})} \right] \right\} \tag{7.24}$$

In terms of the basis given by (6.21), the sesquilinear form (7.11) may be written

$$\langle \psi^{(M^2)}, \phi^{(M^2)} \rangle = \frac{M^2}{2} \sum_{jm} \int d\mu(\mathbf{v}) [\psi_{jm}^{(+)*}(M, \mathbf{v}) \phi_{jm}^{(+)}(M, \mathbf{v}) - \psi_{jm}^{(-)*}(M, \mathbf{v}) \phi_{jm}^{(-)}(M, \mathbf{v})] \tag{7.25}$$

For each of the operators $\mathrm{Op} \in \{X^\mu, P^\mu, L^{\mu\nu}, J^{\mu\nu}, S^{\mu\nu}\}$ [see (6.23)–(6.26)], one may define a corresponding surface functional $\mathrm{Op}\,[\sigma; \psi^*, \psi]$ by[3]

$$\begin{aligned} \mathrm{Op}\,[\sigma; \psi^*, \psi] &= \int dM^2 \langle \psi^{(M^2)}, \mathrm{Op}\, \psi^{(M^2)} \rangle \\ &= \sum_{jm} \int d\mu(M, \mathbf{v}) [\psi_{jm}^{(+)*}(M, \mathbf{v})\, \mathrm{Op}\, \psi_{jm}^{(+)}(M, \mathbf{v}) \\ &\quad - \psi_{jm}^{(-)*}(M, \mathbf{v})\, \mathrm{Op}\, \psi_{jm}^{(-)}(M, \mathbf{v})] \end{aligned} \tag{7.26}$$

For example,

$$P^\mu[\sigma; \psi^*, \psi] = \sum_{jm} \int d\mu(M, \mathbf{v}) M u^\mu(\mathbf{v}) [\psi_{jm}^{(+)*}(M, \mathbf{v}) \psi_{jm}^{(+)}(M, \mathbf{v}) + \psi_{jm}^{(-)*}(M, \mathbf{v}) \psi_{jm}^{(-)}(M, \mathbf{v})] \tag{7.27}$$

One can readily show that

$$[P^\mu[\sigma; \psi^*, \psi], \psi_{jm}(x)]_{PB-} = -\partial^\mu \psi_{jm}(x) \tag{7.28}$$

for $x \in \sigma$, and

$$[P^\mu[\sigma; \psi^*, \psi], P^\nu[\sigma; \psi^*, \psi]]_{PB-} = 0 \tag{7.29}$$

Again an entire set of such relations exists which corresponds to the expressions (6.23)–(6.26) and to the commutation relations (5.7)–(5.9), (5.11), and (5.13).

Using (7.10) and (6.21), the surface functionals (7.26) may be reexpressed in terms of the functions (7.14) and their conjugates. If this is done, the surface functionals (7.26) have the form of integrals of conserved flux densities over a spacelike hypersurface. For the energy-momentum and the orbital, spin, and total angular momenta, the forms of these conserved flux densities are well

[3] A corresponding "matrix element" $\mathrm{Op}\,[\sigma; \psi^*, \phi]$ may also be defined.

known. The position-time functional is given by

$$X^\mu[\sigma; \psi^*, \psi]$$

$$= i \sum_{jm} \int d\mu(M, \mathbf{v})$$

$$\times \left\{ \psi_{jm}^{(+)*}(M, \mathbf{v}) \left[u^\mu(\mathbf{v}) \frac{\partial}{\partial M} - \frac{1}{M} A^{ab}(\mathbf{v}) u^\mu_{,a}(\mathbf{v}) \frac{\partial}{\partial v^b} \right] \psi_{jm}^{(+)}(M, \mathbf{v}) \right.$$

$$\left. + \psi_{jm}^{(-)*}(M, \mathbf{v}) \left[u^\mu(\mathbf{v}) \frac{\partial}{\partial M} - \frac{1}{M} A^{ab}(\mathbf{v}) u^\mu_{,a}(\mathbf{v}) \frac{\partial}{\partial v^b} \right] \psi_{jm}^{(-)}(M, \mathbf{v}) \right\} \tag{7.30}$$

The corresponding conserved flux density is

$$\chi^{\mu\nu}[x] = -i \sum_{jm} \left\{ x^\mu [\psi_{jm}^{(M^2)*}(x) \partial^\nu \psi_{jm}^{(M^2)}(x) - \partial^\nu \psi_{jm}^{(M^2)*}(x) \psi_{jm}^{(M^2)}(x)] \right.$$

$$+ 2 \left[\psi_{jm}^{(M^2)*}(x) \frac{\partial}{\partial M^2} \partial^\mu \partial^\nu \psi_{jm}^{(M^2)}(x) - \partial^\nu \psi_{jm}^{(M^2)*} \frac{\partial}{\partial M^2} \partial^\mu \psi_{jm}^{(M^2)}(x) \right]$$

$$\left. + g^{\mu\nu} \psi_{jm}^{(M^2)*}(x) \psi_{jm}^{(M^2)}(x) \right\} \tag{7.31}$$

and

$$X^\mu[\sigma; \psi^*, \psi] = \int_\sigma dM^2 d\sigma_\nu(x) \chi^{\mu\nu}[x]$$

$$= -i \sum_{jm} \int_t dM^2 d^3\mathbf{x}$$

$$\times \left\{ x^\mu [\psi_{jm}^{(M^2)*}(x) \dot{\psi}_{jm}^{(M^2)}(x) - \dot{\psi}_{jm}^{(M^2)*}(x) \psi_{jm}^{(M^2)}(x)] \right.$$

$$+ 2 \left[\psi_{jm}^{(M^2)*}(x) \frac{\partial}{\partial M^2} \partial^\mu \dot{\psi}_{jm}^{(M^2)}(x) - \dot{\psi}_{jm}^{(M^2)*}(x) \frac{\partial}{\partial M^2} \partial^\mu \psi_{jm}^{(M^2)}(x) \right]$$

$$\left. + g^{\mu 0} \psi_{jm}^{(M^2)*}(x) \psi_{jm}^{(M^2)}(x) \right\} \tag{7.32}$$

Note that in (7.31), $\psi_{jm}^{(M^2)}(x)$ is a function of the five variables M^2 and x^μ, so that the derivatives $\partial/\partial M^2$ and ∂^μ commute. If this point is kept in mind, it is straightforward to verify

$$\partial_\nu \chi^{\mu\nu}[x] = 0 \tag{7.33}$$

In (7.32), $x^0 = t$ is held constant, so that it is only possible to partially integrate with respect to the variables M^2 and $\mathbf{x}$. It is necessary to employ the relation

$$\dot{\psi}_{jm}^{(M^2)}(x) = -(-\nabla^2 + M^2) \psi_{jm}^{(M^2)}(x) \tag{7.34}$$

in the demonstration that (7.32) is a real functional. The integrated terms may be neglected (see Appendix B).

The above analysis has dealt with the case of fields $\psi_{jm}^{(M^2)}(x)$ which depend continuously on the variable M^2. Corresponding formulas for the case of a single discrete mass may be obtained provided the normalization is suitably adjusted. The modifications required for the energy-momentum and the orbital, spin and total angular momentum functionals are minor. For the position-time functional, instead of (7.30), one must use,

$$X^\mu[\sigma; \psi^*, \psi] = -i \sum_{jm} \frac{M^2}{2} \int d\mu(\mathbf{v}) \times \left\{ \psi_{jm}^{(+)*}(M, \mathbf{v}) \frac{1}{M} \left[A^{ab}(\mathbf{v}) u^\mu_{,a}(\mathbf{v}) \frac{\partial}{\partial v^b} + \frac{3}{2} u^\mu(\mathbf{v}) \right] \psi_{jm}^{(+)}(M, \mathbf{v}) + \psi_{jm}^{(-)*}(M, \mathbf{v}) \frac{1}{M} \left[A^{ab}(\mathbf{v}) u^\mu_{,a}(\mathbf{v}) \frac{\partial}{\partial v^b} + \frac{3}{2} u^\mu(\mathbf{v}) \right] \psi_{jm}^{(-)}(M, \mathbf{v}) \right\} \tag{7.35}$$

where the normalization of $\psi_{jm}^{(\pm)}(M, \mathbf{v})$ is given by

$$\frac{M^2}{2} \int d\mu(\mathbf{v}) \sum_{jm} [\psi_{jm}^{(+)*}(M, \mathbf{v}) \psi_{jm}^{(+)}(M, \mathbf{v}) + \psi_{jm}^{(-)*}(M, \mathbf{v}) \psi_{jm}^{(-)}(M, \mathbf{v})] = 1 \tag{7.36}$$

instead of by (6.12). In this connection, it is interesting to note that the eigendifferentials

$$\psi_{jm}^{(M^2, \Delta M^2)}(x) = \frac{1}{(\Delta M^2)^{1/2}} \int_{M^2 - 1/2\Delta M^2}^{M^2 + 1/2\Delta M^2} dM^2 \psi_{jm}^{(M^2)}(x) \tag{7.37}$$

have the customary dimension of length $[L^{-1}]$.

Much of the analysis of Sections (6) and (7) can be carried over to the case of a single spin s. It is a straightforward matter to write down functionals corresponding to (6.27) and (7.26) using the basis $\psi_{s,\lambda}^{(\pm)}(M, \mathbf{v})$ for fixed s. The bracket operations (6.20) and (7.24) are easily restricted to the case of fixed s. Of course, the operator X^μ, $L^{\mu\nu}$, and $S^{\mu\nu}$ and the corresponding functionals will suffer the limitations discussed at the end of Section 5. The momentum space expressions may be reexpressed in terms of the fields $\psi_{jm}^{(s)}(x)$ and $\psi_{jm}^{(M^2, s)}(x)$ defined by dropping the sum over s in (6.8) and (7.2), respectively. The bracket (7.22) is replaced by

$$[\psi_{j'm'}^{(s)}(x'), \psi_{jm}^{(s)*}(x)]_{PB\pm} = \int \frac{dM^2}{2\pi} \Pr^{(s)}\left(\frac{i\partial'}{M}\right) \Delta(x' - x; M^2) \tag{7.38}$$

The projection operator $\Pr^{(s)}(i\partial/M)$ defined in Appendix F is an infinite power series in the derivative ∂^μ. Whether this fact leads to nonlocal behavior is an open question.

8. INADEQUACY OF THE CUSTOMARY FORMALISM

There are two wave-function formalisms which are naturally adapted to the description of events in Minkowski space–time. These were described above in Sections 6 and 7, respectively. The formalism usually employed in relativistic quantum kinematics corresponds to neither of these but rather contains elements of both. In this section it will be shown that although it is, in fact, necessary to use aspects of both formalisms, they are incompatible except for the trivial case of free one-particle states. Then the *ad hoc* "cut and paste" measures required to construct the traditional structure as a blend of the two natural formalisms will be discussed.

Throughout the following discussion it is important for the reader to keep firmly in mind the systematic differences in the corresponding formulas of the two formalisms. In this connection, it should be emphasized that the space–time density description of Section 6 and the hypersurface flux density description of Section 7 are quite distinct, and the systematic differences in corresponding formulas should be kept firmly in mind. In this connection it is useful to compare the sesquilinear forms (6.5), (6.12) and (7.7), (7.11), the orthonormality relations (6.10) and (7.8), the completeness conditions (6.13) and (7.12), the brackets (6.17), (6.20) and (7.21), (7.24), and the operator functionals (6.27) and (7.26). In particular, note that $P^{\mu}[\psi^*, \psi]$, given by (6.28), is not positive definite, while $P^{\mu}[\sigma; \psi^*, \psi]$, given by (7.27), is positive definite. Also, the number functional

$$N[\psi^*, \psi] = \sum_{s\lambda} \int d\mu(M, \mathbf{v})[\psi_{s\lambda}^{(+)*}(M, \mathbf{v})\psi_{s\lambda}^{(+)}(M, \mathbf{v}) + \psi_{s\lambda}^{(-)*}(M, \mathbf{v})\psi_{s\lambda}^{(-)}(M, \mathbf{v})] \tag{8.1}$$

is positive definite, while the charge functional

$$Q[\sigma; \psi^*, \psi] = \sum_{s\lambda} \int d\mu(M, \mathbf{v})[\psi_{s\lambda}^{(+)*}(M, \mathbf{v})\psi_{s\lambda}^{(+)}(M, \mathbf{v}) - \psi_{s\lambda}^{(-)*}(M, \mathbf{v})\psi_{s\lambda}^{(-)}(M, \mathbf{v})] \tag{8.2}$$

is not positive definite.

Of the two formalisms, only the space–time density formalism has a positive-definite inner product, namely, (8.1). Since a positive–definite form is required for the usual interpretation of quantum mechanics, one must use the space–time density formalism. It is interesting to note that an entirely similar formalism may be constructed for any Euclidean or pseudo-Euclidean space. In the case of a pseudo-Euclidean space, provided that all momentum star classes are treated *symmetrically*, no connection between spin and statistics results. In the Euclidean case, this lack of spin-statistics connection follows automatically from the fact that there is only one momentum star class (other

than $\mathbf{p} = \mathbf{o}$). Moreover, if all star classes are included in the completeness relation one obtains an n-dimensional Dirac δ function where n is the total dimension of the space. If only some of the star classes are included, one obtains approximations to this Dirac δ function; for example, the *even* function $\Delta_1(x - y; M^2)$ approximates $\delta^4(x - y)$ for wave functions which have momentum support on the forward or backward mass shell in the case of Minkowski space–time. The energy functional for this formalism is not in general positive definite.

On the other hand, the flux density formalism has a positive-definite energy functional but does *not* have a positive-definite inner product. Rather the best one can do is the indefinite form given by (8.2). The completeness relation for this form has a minus sign associated with the negative frequency states, and it is *this sign* which leads to the *causal* functions $\Delta(x - y; M^2)$. Again, the decoupling of spin and statistics follows from the *symmetric* treatment of positive and negative frequency states. It is the flux density formalism with its conserved flux densities and causal propagators that is required by Lagrangian dynamics.

Thus for relativistic quantum mechanics both the space–time density and the flux density formalisms must be used. But these are incompatible except for the case of free, single-particle states. Compare the momentum space expressions for the respective bracket forms (6.20) and (7.24). It is clear that one may employ the Fock space construction either for (6.20) or for (7.24) but not for both simultaneously; therefore, one is restricted to single particle states. Moreover, if the inner product (6.5) of the flux density formalism is projected onto a spacelike hypersurface, one obtains the result (7.17). By definition, this inner product is time independent for the case of free propagation. However, it is easy to show by considering the interaction Lagrangian for the simple case of a complex scalar field $\phi(x)$

$$\mathscr{L}_{\text{int}} = g(\phi^*(x)\phi(x))^n \tag{8.3}$$

that the inner product (7.17) is not invariant for a general Hamiltonian flow. Note that the nonlocality in the expression (7.17) follows from the fact that a quantity which has a local expression in space–time is projected onto a three-dimensional subspace.

The conclusion to be drawn from the above discussion is that quantum mechanics and Lagrangian dynamics are incompatible in the context of special relativity.

In order to secure this conclusion, it is necessary to show that the formalism traditionally used in relativistic quantum mechanics is not a viable alternative but rather consists of an irrational blend of the space–time density formalism and the flux density formalism.

Recall that the space–time density formalism has a positive-definite

sesquilinear form but an indefinite energy functional and that the situation is reversed for the flux density formalism. For a classical field, the flux density formalism is adequate because the positive-definite inner product is not required, and the positive and negative frequency parts of the field can be treated symmetrically without any difficulty. However, for relativistic quantum mechanics, both the sesquilinear form and the energy functional must be positive definite. This demand leads to the first *ad hoc* step, namely, the reinterpretation of the negative frequency part of the wave function. The *conjugate* of the negative frequency part of the wave function is associated with a particle state; consequently, the expansion

$$\psi_{jm}(x) = \sum_{s\lambda} \int d\mu(M, \mathbf{v})[f_{jm}^{(+)}(x; M, \mathbf{v}, s, \lambda)\psi_{s\lambda}^{(+)}(M, \mathbf{v}) + \tilde{f}_{jm}^{(-)}(x; M, \mathbf{v}, s, \lambda)\tilde{\psi}_{s\lambda}^{(-)*}(M, \mathbf{v})] \tag{8.4}$$

where

$$\tilde{f}_{jm}^{(-)}(x; M, \mathbf{v}, s, \lambda) = \sum_{\lambda'} f_{jm}^{(-)}(x; M, \mathbf{v}, s, \lambda')U_{\lambda'\lambda}^{(s)} \tag{8.5}$$

is used in place of (6.8). The wave function $\tilde{\psi}_{s\lambda}^{(-)}(M, \mathbf{v})$ transforms in the same way as $\psi_{s\lambda}^{(+)}(M, \mathbf{v})$ under an element of the Poincaré group, and $U^{(s)}$ is the standard unitary matrix (Fano and Racah, 1959) which transforms the representation $D^{(s)}$ of 0(3) into the conjugate representation $D^{(s)*}$. Since $U^{(s)}$ is unitary, a sesquilinear functional of the field (8.4) can be obtained from the corresponding sesquilinear functional of the field (6.8) by means of the substitution

$$\begin{aligned} \psi_{s\lambda}^{(-)}(M, \mathbf{v}) &\to \tilde{\psi}_{s\lambda}^{(-)*}(M, \mathbf{v}) \\ \psi_{s\lambda}^{(-)*}(M, \mathbf{v}) &\to \tilde{\psi}_{s\lambda}^{(-)}(M, \mathbf{v}) \end{aligned} \tag{8.6}$$

For example, the *nondiagonal* versions of (8.1) and (8.2) become

$$N[\psi^*, \phi] = \sum_{s\lambda} \int d\mu(M, \mathbf{v})[\psi_{s\lambda}^{(+)*}(M, \mathbf{v})\phi_{s\lambda}^{(+)}(M, \mathbf{v}) + \tilde{\psi}_{s\lambda}^{(-)}(M, \mathbf{v})\tilde{\phi}_{s\lambda}^{(-)*}(M, \mathbf{v})] \tag{8.7}$$

and

$$Q[\sigma, \psi^*, \phi] = \sum_{s\lambda} \int d\mu(M, \mathbf{v})[\psi_{s\lambda}^{(+)*}(M, \mathbf{v})\phi_{s\lambda}^{(+)}(M, v) - \tilde{\psi}_{s\lambda}^{(-)}(M, \mathbf{v})\tilde{\phi}_{s\lambda}^{(-)*}(M, \mathbf{v})] \tag{8.8}$$

It is evident that the conjugation operator * is applied to $\tilde{\phi}^{(-)}$ rather than $\tilde{\psi}^{(-)}$ in (8.7) and (8.8) and that the same will be the case for every sesquilinear functional discussed in Sections 6 and 7. This is the first inconvenient result of applying the apparently harmless reinterpretation principle. This difficulty appears less serious for the case of diagonal functionals for which $\psi = \phi$

because the defect can be remedied by a simple reordering of the wave functions. The situation is similar when field operators are used. This *ad hoc* procedure of changing the order *by hand* is usually called normal ordering.

The substitution (8.6) has a striking effect on the brackets defined by (6.17) and (7.21). Consideration of the momentum space expansions (6.20) and (7.24) shows that (in the *boson* case) the brackets are interchanged under this substitution after a suitable reordering of terms. Thus

$$\{f, g\}_- \leftrightarrow i[f, g]_{PB-} \tag{8.9}$$

Instead of (7.23), one obtains

$$\begin{aligned} i[\tilde{\psi}^{(-)}_{s'\lambda'}(M', \mathbf{v}'), \psi^{(+)*}_{s\lambda}(M, \mathbf{v})]_{PB-} \\ &= \delta_{s's}\delta_{\lambda'\lambda}\delta(M'^2 - M^2)2M\left[\frac{1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})}{M}\right]^3\delta^3(\mathbf{v}' - \mathbf{v}) \\ &= i[\tilde{\psi}^{(-)}_{s'\lambda'}(M', \mathbf{v}'), \tilde{\psi}^{(-)*}_{s\lambda}(M, \mathbf{v})]_{PB-} \end{aligned} \tag{8.10}$$

and the usual Fock construction may be used for multiparticle states in the flux density formalism.

The normal ordered form of (8.7) is

$$\begin{aligned} N[\psi^*, \phi] = \sum_{s\lambda}\int d\mu(M, \mathbf{v})[&\psi^{(+)*}_{s\lambda}(M, \mathbf{v})\phi^{(+)}_{s\lambda}(M, \mathbf{v}) \\ &+ \tilde{\psi}^{(-)*}_{s\lambda}(M, \mathbf{v})\tilde{\phi}^{(-)}_{s\lambda}(M, \mathbf{v})] \end{aligned} \tag{8.11}$$

This expression is used as the inner product in the conventional formalism. In the case of discrete mass, the integration over the mass is of course suppressed. If this expression is projected onto a spacelike hypersurface using (8.4), one obtains the rather complicated formula

$$\begin{aligned} (\psi, \phi) = \tfrac{1}{2}\int dM^2 \int d^3\mathbf{x}d^3\mathbf{y} \sum_{jm} \{&[\dot{\psi}^{(M^2)*}_{jm}(\mathbf{x})\Delta_1(\mathbf{x} - \mathbf{y}; M^2)\phi^{(M^2)}_{jm}(\mathbf{y}) \\ &+ (\nabla_x\psi^{(M^2)}_{jm}(\mathbf{x}))^*\cdot\Delta_1(\mathbf{x} - \mathbf{y}; M^2)(\nabla_y\phi^{(M^2)}_{jm}(\mathbf{y})) \\ &+ M^2\psi^{(M^2)*}_{jm}(\mathbf{x})\Delta_1(\mathbf{x} - \mathbf{y}; M^2)\phi^{(M^2)}_{jm}(\mathbf{y})] \\ &+ [\text{complex conjugate}]\} \\ + \frac{i}{2}\int dM^2 \int d^3\mathbf{x} \sum_{jm} \{&[\dot{\psi}^{(M^2)*}_{jm}(\mathbf{x})\phi^{(M^2)}_{jm}(\mathbf{x}) \\ &- \psi^{(M^2)*}_{jm}(\mathbf{x})\dot{\phi}^{(M^2)}_{jm}(\mathbf{x})] \\ &+ [\text{complex conjugate}]\} \end{aligned} \tag{8.12}$$

where $\Delta_1(\mathbf{x} - \mathbf{y}; M^2)$ is given by (7.18).

The expression (8.12) should be compared and contrasted to (7.17). The nonlocal character of the real part of (8.12) again follows from the fact that

the inner product belongs to the four-dimensional space–time formalism and is projected onto a three-dimensional spacelike hypersurface. In contrast with (7.17), the imaginary part of (8.12) is local and defines a real, nondegenerate symplectic form for the wave functions $[\dot{\psi}_{jm}^{(M^2)}(\mathbf{x}), \psi_{jm}^{(M^2)}(\mathbf{x})]$. This symplectic form and the modified bracket relations (8.10) indicate that the reinterpretation and normal ordering principles succeed in adapting the inner product structure of the space–time density formalism to the Lagrangian structure of the flux density formalism (Bongaarts, 1972). However, in addition to the fact that these two principles are rather arbitrary, not everything works out so nicely.

In the space–time density and flux density formalism, one readily obtains the respective bracket relations

$$\{X^\mu(\psi^*, \psi), P^\nu(\psi^*, \psi)\}_- = ig^{\mu\nu}N(\psi^*, \psi) \tag{8.13}$$

and

$$[X^\mu(\sigma; \psi^*, \psi), P^\nu(\sigma; \psi^*, \psi)]_{PB-} = g^{\mu\nu}Q(\sigma; \psi^*, \psi) . \tag{8.14}$$

These bracket relations are sensible because $N(\psi^*, \psi)$ and $Q(\sigma; \psi^*, \psi)$ given by (8.1) and (8.2) are just the respective *unit* functionals in the two formalisms. However, after the principles of reinterpretation and normal ordering have been applied to the flux density formalism, one finds that the relation (8.14) still holds. Unfortunately, $Q(\sigma; \psi^*, \psi)$ is no longer the *unit* functional for the resulting formalism. This role is now played by $N(\psi^*, \psi)$. Consequently, in the conventional formalism, the position-time functionals do not transform properly under translations. This failure is now perfectly understandable and is an indication that the application of the principles of reinterpretation and normal ordering is not a consistent procedure.

Since the conventional formalism does not treat the positive and negative frequency parts of the wave function in a symmetric way, the choice of statistics is no longer arbitrary. For wave functions which transform according to a unitary representation of 0(1, 3), one must choose Bose–Einstein statistics. By using finite-dimensional representations of 0(1, 3), one obtains an additional sign $(-)^{2s}$, where s is the particle's spin, between the positive and negative frequency contributions to the bracket relation for the wave function; consequently, one can obtain causal bracket relations provided the customary relation between spin and statistics is adopted. This switch to finite-dimensional representations leads to further difficulties with position-time operators since only truncated versions of such operators can be defined unless unitary representations of 0(1, 3) are used (see the comments at the end of Section 5).

The irreducible, finite-dimensional representations of 0(1, 3) are given by (Naimark, 1964)

$$D^{(A,B)} = D^{(A,0)} \otimes D^{(0,B)} \tag{8.15}$$

where $D^{(A,0)}$ and $D^{(0,B)}$ are complex extensions of the UIR of 0(3) and $A, B \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\}$. The quantum numbers A and B are usually suppressed by assigning them more or less arbitrary values possibly chosen for simplicity. These additional, unwanted quantum numbers result from the fact that the formalism is adapted to the description of a system with three degrees of freedom in excess of those possessed by a classical particle. This point is discussed further in Section 10. The same difficulty arises in connection with wave functions which transform according to a UIR of 0(1, 3), in which case the additional quantum numbers are (ρ, k_0). In *this* particular, none of the three formalisms is adequate.

The use of finite-dimensional spinor wave functions involves additional difficulties. For example, the inner product for the four-component spinor wave function $\psi(p)$ which transforms according to $D^{(1/2,0)} \oplus D^{(0,1/2)}$ is given by

$$(\psi, \phi) = \int \frac{d^3\mathbf{p}}{2\omega_p} \psi^*(p) \frac{\gamma^0 \gamma \cdot p}{M} \phi(p) \tag{8.16}$$

The presence of the Hermitian Hilbert space metric matrix $(\gamma^0\gamma \cdot p)/M$ is due to the fact that the spin projection operator is not diagonal in the chosen basis. The fact that this matrix depends on p^μ is inconvenient from the viewpoint of constructing a Lagrangian; consequently, it is customary to discard half the degrees of freedom by projecting onto a subspace for which

$$\frac{\gamma \cdot p}{M} \psi(p) = \psi(p) \tag{8.17}$$

This relation is just the Dirac equation. For particles with higher spin s, the metric matrix will be at least of degree $2s$ in p^μ and the manipulations required to remove the momentum dependence of the metric matrix are considerably more complicated.

Another point of interest is the fact that in the space–time density and the flux density formalisms, both the completeness condition and the bracket relation for the basic fields yield the *unit* distribution for the system in question. Compare (6.13) with (6.18) and (7.12) with (7.22). However, in the conventional formalism no such relationship holds. This peculiarity of the conventional formalism bears a strong resemblance to the difficulty that arises in connection with the position-time functional where the problem is the *unit* functional. [See the comments made above in connection with (8.13) and (8.14).]

In conclusion, the customary kinematic formalism used as a foundation for relativistic quantum mechanics and relativistic quantum field theory contains a number of anomalies which can be readily understood provided that this formalism is viewed as an *ad hoc* blend of the space–time density

and the flux density formalisms described in Sections 6 and 7. Although each of these formalisms is self-consistent, they are incompatible except for the trivial case of free single-particle states. The *ad hoc* adaptations required to construct a workable formalism are only partially successful, and the resulting customary formalism is internally inconsistent.

It should be emphasized that the difficulties discussed above do not apply in the case of classical fields. First, in the classical case, a positive-definite inner product is not required and one can simply use the flux density formalism outlined in Section 7. Second, the tensor indices on classical fields have a different meaning from that required by the particle interpretation associated with the quantum case.

9. POSITION OPERATORS FOR THE EUCLIDEAN CASE

The definition and construction of position operators in nonrelativistic Schrödinger mechanics is well known. However, it is not customary to take as the starting point of the discussion the unitary irreducible representations (UIRs) of the Euclidean group $E(3)$. In the preceding sections, the construction of position-time operators starts with the UIRs of the Poincaré group $E(1, 3)$; therefore, it is useful to review here the parallel construction for the Euclidean case. Since the formulas for the Minkowski case are justified in detail above, only an outline of the Euclidean case is presented here. It should be emphasized that, although formulas are given only for the group $E(3)$, the construction may be carried through for the Euclidean group in n dimensions, $E(n)$.

Denote by $x_i(P)$ the coordinates of a point P in Euclidean space. Under an element $(a, R(\omega)) \in E(3)$, the coordinates transform according to

$$x' = a + Rx \tag{9.1}$$

The rotation matrix is given by

$$R(\omega) = \exp\left[\frac{i}{2}\omega_{ij}I_{ij}\right] \tag{9.2}$$

where

$$(I_{ij})_{kl} = -i(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \tag{9.3}$$

A UIR of $E(3)$ has Hermitian generators P_i and J_{ij} for translations and rotations, respectively. For a complete set of commuting observables, one may choose the three-momentum and helicity operators, namely,

$$P_i \qquad \tfrac{1}{2}e_{ijk}J_{ij}P_k \tag{9.4}$$

which have eigenvalues p_i and $|\mathbf{p}|\lambda$, where λ denotes helicity. The momentum

space wave function for a particle with helicity λ is a complex-valued function $\psi_\lambda(\mathbf{p})$ square integrable with respect to the inner product

$$(\psi, \phi) = \int d^3\mathbf{p}\psi_\lambda(\mathbf{p})\phi_\lambda(\mathbf{p}) \tag{9.5}$$

Under a rotation, the wave functions transform according to

$$[U(R)\psi]_\lambda(\mathbf{p}) = \exp\{i\lambda\theta[R^{-1}(\mathbf{p})RR(R^{-1}\mathbf{p})]\}\psi_\lambda(R^{-1}\mathbf{p}) \tag{9.6}$$

where $R(\mathbf{p})$ is the rotation in the plane of $\hat{p} = (0, 0, |\mathbf{p}|)$ and $\mathbf{p}$ which takes $\hat{p}$ into $\mathbf{p}$.

Wave functions which transform according to (9.6) are intrinsic tensors on the momentum sphere. Introduce the coordinates (p, ξ^1, ξ^2) by

$$p_i = \frac{p}{1 + \frac{1}{4}(\xi\cdot\xi)}[\xi^1, \xi^2, 1 - \tfrac{1}{4}(\xi\cdot\xi)] = pu_i(\xi) \tag{9.7}$$

then

$$\begin{aligned} dp_i &= u_i(\xi)\, dp + pu_{i,\alpha}(\xi)\, d\xi^\alpha \\ d\mathbf{p}\cdot d\mathbf{p} &= (dp)^2 + \left[\frac{p}{1 + \frac{1}{4}(\xi\cdot\xi)}\right]^2 d\xi\cdot d\xi \\ u_{i,\alpha}(\xi) &= \frac{\partial u_i(\xi)}{\partial\xi^\alpha} \end{aligned} \tag{9.8}$$

Under a rotation R, an intrinsic vector field ψ defined on the sphere transforms according to

$$[U(R)\psi]_\alpha(p, \xi) = \frac{\partial(R^{-1}\xi)^\beta}{\partial\xi^\alpha}\psi_\beta(p, R^{-1}\xi) \tag{9.9}$$

The transformation law (9.9) is the same as that given by (9.6) for $\lambda = \pm 1$. The difference lies in the fact that the tensor indices in (9.9) refer to the coordinate basis $u_{i,\alpha}(\xi)$, while the helicity $\lambda = \pm 1$ in (8.6) refers to the basis

$$\frac{n_{i1}(\xi) \pm n_{i2}(\xi)}{2^{1/2}}$$

where

$$\begin{aligned} n_{i\alpha}(\xi) &= [1 + \tfrac{1}{4}(\xi\cdot\xi)]u_{i,\alpha}(\xi) \\ n_\alpha(\xi)\cdot n_\beta(\xi) &= \delta_{\alpha\beta} \end{aligned} \tag{9.10}$$

Using this correspondence parallel transport can be defined for the field $\psi_\lambda(p, \xi)$ in a manner analogous to that used in the Minkowski case (see Section 4).

For a scalar wave function $\psi(p, \xi)$, the usual definition of the position operator X_i is

$$X_i = i\frac{\partial}{\partial p_i} = i\left[u_i(\xi)\frac{\partial}{\partial p} + \frac{1}{p}A^{\alpha\beta}(\xi)u_{i,\alpha}(\xi)\frac{\partial}{\partial \xi^\beta}\right] \tag{9.11}$$

where $A^{\alpha\beta}(\xi) = [1 + \frac{1}{4}(\xi\cdot\xi)]^2\delta_{\alpha\beta}$ is the metric tensor on the sphere. In the case of nonzero helicity, the expression (9.11) must be modified by replacing the partial derivative $\partial/\partial\xi^\beta$ by the covariant derivative $\delta/\delta\xi^\beta$, so that

$$X_i = i\left[u_i(\xi)\frac{\partial}{\partial p} + \frac{1}{p}A^{\alpha\beta}(\xi)u_{i,\alpha}(\xi)\frac{\delta}{\delta\xi^\beta}\right] \tag{9.12}$$

Compare (9.12) with (2.7). The operators defined by (9.12) satisfy

$$[X_i, P_j] = i\delta_{ij} \tag{9.13}$$

However,

$$[X_i, X_j] \neq 0 \tag{9.14}$$

because the covariant derivatives do not commute. If L_{ij} is defined by

$$L_{ij} = X_iP_j - X_jP_i \tag{9.15}$$

then

$$J_{ij} = L_{ij} + S_{ij} \tag{9.16}$$

However, L_{ij} and S_{ij} do not satisfy the same commutation relations as J_{ij} and do not mutually commute.

The standard solution of these difficulties is to use wave functions which transform under rotations according to a UIR of 0(3), the Casimir operator of which then represents the particle's total spin angular momentum. Except for the scalar case, the wave function must then describe more than a single helicity state; for example, $\lambda = 0, \pm 1$ for the case of spin 1. In a Cartesian basis the position operator is just the partial derivative (9.11). One obtains the relation (9.16) where the operators L_{ij} are again defined by (9.15) and the operators S_{ij} are just the generators of the UIR of 0(3). These operators, L_{ij} and S_{ij}, are Hermitian, mutually commute, and satisfy the same commutation relations as J_{ij}. Since

$$[X_i, \tfrac{1}{2}e_{jkl}J_{jk}P_l] = [X_i, \tfrac{1}{2}e_{jkl}S_{jk}P_l] \neq 0 \tag{9.17}$$

the position operator has nonzero matrix elements between states of different helicity. A similar comment applies to the operators L_{ij} and S_{ij}. If these operators are restricted to a space of definite $|\lambda|$ by transforming from a Cartesian basis to a spherical basis and discarding the unwanted terms, the

operators are truncated, and it is this truncation which leads to the unacceptable commutation relations for the position operator (9.12) and for the corresponding truncated versions of the operators L_{ij} and S_{ij}, namely,

$$L_{ij} = iA^{\alpha\beta}(\xi)[u_i(\xi)u_{j,\alpha}(\xi) - u_j(\xi)u_{i,\alpha}(\xi)]\frac{\delta}{\delta\xi^\beta} \tag{9.18}$$

and

$$(S_{ij})_{\alpha\beta} = i[u_{i,\alpha}(\xi)u_{j,\beta}(\xi) - u_{j,\alpha}(\xi)u_{i,\beta}(\xi)] \tag{9.19}$$

An important difference between the Euclidean and the Minkowski cases is the fact that all of the UIR's of 0(1, 3) are infinite dimensional so that wave functions which represent an infinite spin tower must be used.

10. CLASSICAL PARTICLE WITH SPIN

The source of the difficulties described in the preceding sections may be traced to the fact that the mathematical formalism of relativistic quantum kinematics is not appropriately adapted to the description of certain aspects of the concept of a point particle with spin. In order to emphasize those features which are not appropriately described, the classical description of the motion of such particles will be briefly reviewed using the formalism of mass shell tensors.

The history of a moving point particle is represented in Minkowski space–time by the world line of the particle, a curve $z^\mu(\mathscr{T})$, where $\mathscr{T}$ denotes the proper time of the particle. Using the velocity coordinates defined by (A.22), one obtains for the unit timelike velocity four vector

$$\frac{dz^\mu}{d\mathscr{T}} = u^\mu(\mathbf{v}(\mathscr{T})) \tag{10.1}$$

Since the force four-vector F^μ is always orthogonal to the velocity four-vector, define f^a by

$$F^\mu = f^a u^\mu_{,a}(\mathbf{v}(\mathscr{T})) \tag{10.2}$$

Then, if M denotes the rest mass of the particle, the equation of motion is

$$M\frac{d\mathbf{v}^a}{d\mathscr{T}} = f^a \tag{10.3}$$

For a particle with charge e in an electromagnetic field $F^{\mu\nu}$ where

$$\begin{aligned} F^{\mu\nu} = &-[u^\mu(\mathbf{v})u^\nu_{,a}(\mathbf{v}) - u^\nu(\mathbf{v})u^\mu_{,a}(\mathbf{v})]\mathscr{E}^a \\ &- \tfrac{1}{2}[u^\mu_{,a}(\mathbf{v})u^\nu_{,b}(\mathbf{v}) - u^\nu_{,a}(\mathbf{v})u^\mu_{,b}(\mathbf{v})]\mathscr{B}^{ab} \end{aligned} \tag{10.4}$$

the mass shell force vector f^a is given by

$$f^a = e\mathscr{E}^a \tag{10.5}$$

A completely adequate model for intrinsic spin does not yet exist; however, it is customary to represent the intrinsic angular momentum of a point particle by a Lorentz four-pseudovector S^μ which is orthogonal to the four velocity at every point of the particle's world line. Even if there is no applied torque, the spin pseudovector S^μ must nevertheless change with time in order to remain orthogonal to the four velocity. If it is also required that $g_{\mu\nu}S^\mu S^\nu$ remain constant along the world line, the spin pseudovector must satisfy the equation for Fermi–Walker transport (Misner et al., 1973)

$$\frac{dS^\mu}{d\mathscr{T}} + g_{\nu\sigma}(u^\mu a^\nu - u^\nu a^\mu)S^\sigma = 0 \tag{10.6}$$

where

$$a^\mu \equiv \frac{d^2z^\mu}{d\mathscr{T}^2} = \frac{du^\mu}{d\mathscr{T}} = u^\mu_{,a}(\mathbf{v})\frac{dv^a}{d\mathscr{T}} \tag{10.7}$$

Define the mass shell spin pseudovector $\mathscr{S}^a$ by

$$S^\mu = \mathscr{S}^a u^\mu_{,a}(\mathbf{v}) \tag{10.8}$$

Then, it is straightforward to show that $\mathscr{S}^a$ satisfies the equation for parallel transport relative to the intrinsic geometry of the mass shell, namely,

$$\frac{d\mathscr{S}^a}{d\mathscr{T}} + \begin{Bmatrix} a \\ b\ c \end{Bmatrix}\mathscr{S}^b\frac{dv^c}{d\mathscr{T}} = 0 \tag{10.9}$$

The expression for Thomas precession (see Misner et al., 1973, p. 175, for an alternate treatment) may be derived in a particularly elegant manner from equation (10.9). This computation is presented in Appendix H.

Higher-order structure of a particle may similarly be described by mass shell tensors; for example, a quadrupole moment is represented by a mass shell tensor $\mathscr{Q}^{ab}$ which is symmetric and traceless:

$$\mathscr{Q}^{ab} = \mathscr{Q}^{ba} \tag{10.10}$$

$$A_{ab}(\mathbf{v})\mathscr{Q}^{ab} = 0$$

and satisfies

$$\frac{d\mathscr{Q}^{ab}}{d\mathscr{T}} + \begin{Bmatrix} a \\ c\ d \end{Bmatrix}\mathscr{Q}^{cb}\frac{dv^d}{d\mathscr{T}} + \begin{Bmatrix} b \\ c\ d \end{Bmatrix}\mathscr{Q}^{ac}\frac{dv^d}{d\mathscr{T}} = 0 \tag{10.11}$$

in the absence of coupling to external fields.

For a particle with a gyromagnetic ratio Γ, equation (10.9) must be replaced by

$$\frac{d\mathscr{S}^a}{d\mathscr{T}} + \left\{ \begin{matrix} a \\ b\ c \end{matrix} \right\} \mathscr{S}^b \frac{dv^c}{d\mathscr{T}} = \Gamma A^{ab}(\mathbf{v}) \epsilon_{bcd} \mathscr{S}^c \mathscr{B}^d \tag{10.12}$$

where

$$\mathscr{B}^{ab} = A_{cd}(\mathbf{v}) \epsilon^{abc} \mathscr{B}^d \tag{10.13}$$

Finally, it is interesting to note that the Abraham four vector Γ^μ is given by (Rohrlich, 1965)

$$\begin{aligned} \Gamma^\mu &= \frac{2}{3} e^2 \left(\frac{da^\mu}{d\mathscr{T}} + a^\lambda a_\lambda u^\mu \right) \\ &= \frac{2}{3} e^2 \left(\frac{d^2 v^a}{d\mathscr{T}^2} + \left\{ \begin{matrix} a \\ b\ c \end{matrix} \right\} \frac{dv^b}{d\mathscr{T}} \frac{dv^c}{d\mathscr{T}} \right) u^\mu_{,a}(\mathbf{v}) \end{aligned} \tag{10.14}$$

The usual heuristic model of an extended particle is a tube of world lines which is everywhere timelike and which has a *small* three-dimensional cross-section at every point along its length. (For a covariant description of such a tube, see Sorg, 1974; Synge, 1974.) Such a tube is most aptly characterized by describing its cross-section at each point along its length. The structural features of each cross-section may be specified by means of tensors belonging to the three space of the cross-section; consequently, in the limit of a point particle, structural features are described by mass shell tensors which vary with the particle's proper time.

It is clear that the spin of a particle and all of its multipole moments are well defined only if the four momentum of the particle is precisely known. In classical physics, this fact does not cause any difficulty since position and momentum may be simultaneously determined with arbitrary precision. However, in quantum mechanics, to the extent that the particle's position is defined, its momentum is not defined, and its spin and multipole moments are not defined either. This fact is the basic source of the difficulties described in the preceding sections.

In principle, the Feynman path integral formalism (Feynman and Hibbs, 1965) for quantum mechanics offers a way out of this difficulty because probability amplitudes are assigned to entire world lines. However, as soon as an attempt is made to define the customary wave function for a particle at a point at a given time, the difficulties with spin described above return. This wave function is defined to be the sum of the probability amplitudes for all the world lines which lead to the space–time point from the past. Clearly, at the space–time point, the various world lines will have different four momenta, so that, unless one is willing to merely average over the spin variables before summing over the world lines, one has difficulty in representing the

spin information. Nevertheless, the difficulties with spin are not built into the foundations of the Feynman path integral formalism, and although the formalism is plagued with other problems, research on this formalism may provide a solution to the problems with spin described in this paper.

There are two essentially different geometries possible for a particle in Minkowski space–time which will be called the sphere geometry and the tube geometry. In both cases, the particle's path in space–time is specified by an everywhere timelike world line. In the sphere geometry, the orientation at each point of the world line is specified by assigning an *arbitrary* tetrad subject only to the condition that the tetrad vary continuously along the world line. In the tube geometry, discussed above in this section, the tetrad at each point of the world line is further restricted by the requirement that the timelike member of the tetrad be tangent to the world line at the given point. In the case of Euclidean three space, the same two alternatives are available. Given a twisted space curve, in the sphere geometry, one may specify an arbitrary triad at each point on it, while for the tube geometry one of the triad's vectors must always be tangent to the space curve at the given point. Now in Section 9, it was shown that satisfactory position operators for the Euclidean case cannot be defined for wave functions which transform according to a UIR of $E(3)$. It is necessary to use wave functions whose spin indices transform according to a UIR of 0(3) rather than according to a UIR of the little group which is 0(2). The wave functions are then described by an additional quantum number, the total intrinsic angular momentum, the presence of which indicates that the wave functions describe three-dimensional objects moving in three-dimensional space. Similarly, in Section 5, it was shown that fully adequate space–time operators for the Minkowski case can be constructed only if wave functions with spin indices which transform according to a UIR of 0(1, 3) are used. In both cases, one is dealing with the sphere geometry rather than the tube geometry. In nonrelativistic physics time is separate from three space and one is describing three-dimensional objects in three space, so that, the sphere geometry is the appropriate one. However, in relativistic physics, space and time are unified and it is necessary to deal with the tube geometry rather than the sphere geometry; consequently, wave functions which transform according to a representation of 0(1, 3), whether finite or infinite dimensional, are not appropriate. Moreover, it is not at all clear that an appropriate quantum description of tube geometry even exists. In the classical case the problem is readily solved by means of mass shell tensors.

The important point to be emphasized here is that simple generalizations of the formalism used in the nonrelativistic case, such as replacement of three tensors by four tensors, are not always appropriate. While it is possible to regard density and flux as components of a four vector or energy and three

momentum as components of four momentum, quantities such as intrinsic spin and multipole moments must be treated quite differently.

11. CONCLUSIONS AND COMMENTS

The kinematic formalism currently used as the foundation for relativistic quantum mechanics and relativistic quantum field theory is not internally consistent and is not appropriate for the description of particles possessing internal structure such as spin and multipole moments. Three main arguments were advanced in support of this conclusion.

First, as discussed in Section 10, the description of particle structure is rather more complicated in the Minkowski case than in the Euclidean case. In the nonrelativistic case, time and space are separate, and unconstrained Euclidean tensors may be used to describe particle structure. In the relativistic case, however, the structural features of a particle must be described in the rest system of the particle, and the Minkowski tensors used for the description are constrained to have nonzero projections only on the three-dimensional subspace orthogonal to the particle's four momentum. Thus, in the relativistic quantum case, any uncertainty in the direction of the four momentum automatically results in an uncertainty in any quantity describing the particle's structure. The use of wave functions which transform according to representations of $0(1, 3)$ allows for unneeded degrees of freedom. The additional quantum numbers, the eigenvalues of the Casimir operators of $0(1, 3)$, must then be arbitrarily specified, and even then additional constraint equations are required.

Even if these objections are brushed aside and wave functions which transform according to representations of $0(1, 3)$ are allowed, there are strong reasons for preferring the infinite-dimensional UIR of $0(1, 3)$ to the finite-dimensional, nonunitary spinor representations of $0(1, 3)$ that are used in the customary formalism. Briefly, there is a completely general construction for position or position-time operators for any Euclidean or pseudo-Euclidean space regardless of signature or dimension (see Sections 2–5 and 9). This construction is based on the covariant momentum derivatives of the momentum space wave functions with respect to the intrinsic geometries of the various momentum space "mass shells" involved. In the Minkowski case, the use of this construction leads to the requirement that wave functions transforming according to a UIR of $0(1, 3)$ be used; consequently, spinor fields must be rejected. Since this construction is accepted as standard in the Euclidean case and since it is quite generally applicable, its rejection in the particular case of Minkowski space–time must be regarded as *ad hoc*.

The third argument showed that the numerous peculiarities of the customary kinematic formalism can be understood provided that this formalism is regarded as an *ad hoc* blend of two other formalisms, the space–time density formalism and the flux density formalism. The latter two formalisms were discussed in Sections 6 and 7, respectively, while the analysis and criticism of the customary formalism was presented in Section 8. The arguments of Section 8 can only be understood if the analogous formulas of Sections 6 and 7 are thoroughly compared and contrasted with each other and with the corresponding formulas of the customary formalism which may be found in any standard text on relativistic quantum mechanics or relativistic quantum field theory. Although the space–time density and the flux density formalisms are themselves internally consistent, their mixture, the customary formalism, is not.

Quantum electrodynamics has successfully passed every experimental test so far devised. Yet the theory is logically flawed. An ingenious recipe has been found for obtaining correct answers from an incorrect theory. At present, it appears likely that experiment will not be able to provide clues to guide research on this difficult but fundamental problem. The relevance of the analysis presented above is that it does point to several specific features of the theory that require radical revison. It should be stressed that the well-known difficulties of relativistic quantum field theory result not merely from the use of inadequate computation methods such as perturbation theory, nor even from the failure to find the precise interaction for which the singularities will cancel. Rather, their origin lies much deeper in the structure of relativistic quantum kinematics itself. It is possible that somewhat different formulations of relativistic quantum mechanics, such as the Feynman path integral formalism briefly discussed in Section 10, will provide a solution to some of the difficulties discussed above; however, it is important to ensure that these difficulties do not merely reappear in a disguised form. On the other hand, the difficulty of constructing a consistent relativistic quantum mechanics appears sufficiently great to justify attempts to find a classical explanation for quantum phenomena in the form of nonlocal hidden-variable theories.

APPENDIX A: THE INTRINSIC GEOMETRIES OF THE INTERIOR OF THE FORWARD LIGHT CONE AND OF THE MASS HYPERBOLOID

In this appendix the notational conventions are fixed and a number of geometric formulas used in the main text are presented. (See any standard text on differential geometry. An elementary treatment is given in McConnell, 1957.)

Let X denote a point in space–time and denote by

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, x^j) = (t, \mathbf{x}) \tag{A.1}$$

the coordinates of X in a given frame. The Minkowski metric is given by

$$[g^{\mu\nu}] = [g_{\mu\nu}] = \text{diag}\,(1, -1, -1, -1) \tag{A.2}$$

The space–time interval is

$$(dX)^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = (dt)^2 - d\mathbf{x}\cdot d\mathbf{x} \tag{A.3}$$

and the measure on space–time is

$$d\mu(x) = d^4x = dx^0\, dx^1\, dx^2\, dx^3 = dt\, d^3\mathbf{x} \tag{A.4}$$

$\{\kappa, \lambda, \mu, \nu, \ldots\}$ are used for space–time indices, while $\{i, j, k, l, \ldots\}$ are used for space indices.

Let P denote a point in the momentum-energy manifold, the interior of the forward light cone in momentum space. Such a point corresponds to a Minkowski four-vector p^μ which satisfies

$$g_{\mu\nu}p^\mu p^\nu > 0 \tag{A.5}$$

However, it is more convenient to use the orthogonal, curvilinear coordinates

$$v^\alpha = (v^0, v^1, v^2, v^3) = (M, v^a) = (M, \mathbf{v}) \tag{A.6}$$

defined by

$$p^\mu = \frac{v^0}{1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})}\left[1 + \frac{\mathbf{v}\cdot\mathbf{v}}{4}, \mathbf{v}\right] \tag{A.7}$$

where

$$p^0 > 0 \qquad M > 0 \qquad |\mathbf{v}| < 2 \tag{A.8}$$

$\{\alpha, \beta, \gamma, \delta, \ldots\}$ are used for velocity-mass indices, while $\{a, b, c, d, \ldots\}$ are used for velocity indices.

It is interesting to note that in the nonrelativistic approximation $|\mathbf{v}| \ll 2$,

$$p^\mu \doteq [M + \tfrac{1}{2}M\mathbf{v}\cdot\mathbf{v}, M\mathbf{v}] \tag{A.9}$$

The differential element in momentum-energy space is given by

$$dp^\mu = p^\mu_{,\alpha}\, dv^\alpha \qquad p^\mu_{,\alpha} \equiv \frac{\partial p^\mu}{\partial v^\alpha} \tag{A.10}$$

where

$$
\begin{aligned}
p^{\mu}_{,0} &= \frac{1}{1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})}\left[1 + \frac{\mathbf{v}\cdot\mathbf{v}}{4}, \mathbf{v}\right] \\
p^{\mu}_{,1} &= \frac{v^0}{[1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})]^2}\left[v^1, \frac{v^1v^1}{2} + \left(1 - \frac{\mathbf{v}\cdot\mathbf{v}}{4}\right), \frac{v^2v^1}{2}, \frac{v^3v^1}{2}\right] \\
p^{\mu}_{,2} &= \frac{v^0}{[1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})]^2}\left[v^2, \frac{v^1v^2}{2}, \frac{v^2v^2}{2} + \left(1 - \frac{\mathbf{v}\cdot\mathbf{v}}{4}\right), \frac{v^3v^2}{2}\right] \\
p^{\mu}_{,3} &= \frac{v^0}{[1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})]^2}\left[v^3, \frac{v^1v^3}{2}, \frac{v^2v^3}{2}, \frac{v^3v^3}{2} + \left(1 - \frac{\mathbf{v}\cdot\mathbf{v}}{4}\right)\right]
\end{aligned}
\tag{A.11}
$$

Define $\eta_{\alpha\beta}$ by

$$\eta_{\alpha\beta} = g_{\mu\nu}p^{\mu}_{,\alpha}p^{\nu}_{,\beta} \tag{A.12}$$

and $\eta^{\alpha\beta}$ by

$$\eta^{\alpha\beta}\eta_{\beta\gamma} = \delta_{\gamma}{}^{\alpha} \tag{A.13}$$

then

$$g^{\mu\nu} = \eta^{\alpha\beta}p^{\mu}_{,\alpha}p^{\nu}_{,\beta} \tag{A.14}$$

Explicitly, $\eta_{\alpha\beta}$ and $\eta^{\alpha\beta}$ are given by

$$[\eta_{\alpha\beta}] = \operatorname{diag}\left\{1, -\frac{(v^0)^2}{[1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})]^2}, -\frac{(v^0)^2}{[1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})]^2}, -\frac{(v^0)^2}{[1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})]^2}\right\} \tag{A.15}$$

$$[\eta^{\alpha\beta}] = \operatorname{diag}\left\{1, -\frac{[1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})]^2}{(v^0)^2}, -\frac{[1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})]^2}{(v^0)^2}, -\frac{[1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})]^2}{(v^0)^2}\right\}$$

The momentum-energy interval is

$$(dP)^2 = \eta_{\alpha\beta}\,dv^{\alpha}\,dv^{\beta} = (dM)^2 - \left[\frac{M}{1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})}\right]^2 d\mathbf{v}\cdot d\mathbf{v} \tag{A.16}$$

The measure on the momentum-energy manifold is

$$d\mu(P) = \left[\frac{M}{1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})}\right]^3 dM\,d^3\mathbf{v} \tag{A.17}$$

Define

$$\eta = \det[\eta_{\alpha\beta}] \tag{A.18}$$

then

$$d\mu(P) = (-\eta)^{1/2}\,d^4v \tag{A.19}$$

The momentum-energy gradient is given by

$$\frac{\partial}{\partial p_\mu} = \eta^{\alpha\beta} p^\mu_{,a} \frac{\partial}{\partial v^\beta} \tag{A.20}$$

Only invariance under the Lorentz group rather than invariance under the larger group (semigroup) which leaves the form (A.16) invariant is required. Each of the mass hyperboloids, which are all similar, is invariant under Lorentz transformations. Some useful results relating to the intrinsic geometry of the forward unit mass hyperboloid follow.

Let U be a point on the forward unit mass hyperboloid with coordinates

$$(v^1, v^2, v^3) = (\mathbf{v}) = (v^a) \tag{A.21}$$

defined by

$$u^\mu(\mathbf{v}) = \frac{1}{1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})}\left[1 + \frac{\mathbf{v}\cdot\mathbf{v}}{4}, \mathbf{v}\right] \tag{A.22}$$

The differential element is given by

$$du^\mu = u^\mu_{,a}\, dv^a \qquad u^\mu_{,a} \equiv \frac{\partial u^\mu}{\partial v^a} \tag{A.23}$$

where

$$p^\mu_{,a} = M u^\mu_{,a} \qquad (a = 1, 2, 3) \tag{A.24}$$

and the $p^\mu_{,a}$ $(a = 1, 2, 3)$ are given explicitly by (A.11).

Define A_{ab} by

$$A_{ab} = -g_{\mu\nu} u^\mu_{,a} u^\nu_{,b} \tag{A.25}$$

and A^{ab} by

$$A^{ab} A_{bc} = \delta_c{}^a \tag{A.26}$$

then

$$A^{ab} u^\mu_{,a} u^\nu_{,b} = u^\mu u^\nu - g^{\mu\nu} \tag{A.27}$$

The hyperboloid metric tensors are explicitly given by

$$\begin{aligned} A_{ab} &= \frac{1}{[1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})]^2}\,\delta_{ab} \\ A^{ab} &= [1 - \tfrac{1}{4}(\mathbf{v}\cdot\mathbf{v})]^2 \delta^{ab} \end{aligned} \tag{A.28}$$

The interval is

$$(dU)^2 = A_{ab}\, dv^a\, dv^b = \frac{1}{[1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})]^2}\, d\mathbf{v}\cdot d\mathbf{v} \tag{A.29}$$

Define

$$A = \det(A_{ab}) \tag{A.30}$$

then the invariant measure is

$$d\mu(U) = (A)^{1/2}\, dv^1\, dv^2\, dv^3 = \frac{d^3\mathbf{v}}{[1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})]^3} \tag{A.31}$$

The Christoffel symbols are defined by

$$\begin{aligned} [ab, c] &= \frac{1}{2}\left(\frac{\partial A_{bc}}{\partial v^a} + \frac{\partial A_{ca}}{\partial v^b} - \frac{\partial A_{ab}}{\partial v^c}\right) \\ \left\{ \begin{matrix} c \\ a\ b \end{matrix} \right\} &= A^{cd}[ab, d] \end{aligned} \tag{A.32}$$

and are given explicitly by

$$\begin{aligned} [ab, c] &= \frac{1}{2}\frac{1}{[1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})]^3}(v^a\delta_{bc} + v^b\delta_{ca} - v^c\delta_{ab}) \\ \left\{ \begin{matrix} c \\ a\ b \end{matrix} \right\} &= \frac{1}{2}\frac{1}{1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})}(v^a\delta_{bc} + v^b\delta_{ca} - v^c\delta_{ab}) \end{aligned} \tag{A.33}$$

The Riemann–Christoffel tensor, defined by

$$R^a_{\cdot bcd} = \frac{\partial}{\partial v^c}\left\{ \begin{matrix} a \\ b\ d \end{matrix} \right\} - \frac{\partial}{\partial v^d}\left\{ \begin{matrix} a \\ b\ c \end{matrix} \right\} + \left\{ \begin{matrix} e \\ b\ d \end{matrix} \right\}\left\{ \begin{matrix} a \\ e\ c \end{matrix} \right\} - \left\{ \begin{matrix} e \\ b\ c \end{matrix} \right\}\left\{ \begin{matrix} a \\ e\ d \end{matrix} \right\} \tag{A.34}$$

is just

$$R_{abcd} = \frac{1}{[1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})]^4}(\delta_{bc}\delta_{da} - \delta_{ca}\delta_{bd}) = A_{ad}A_{bc} - A_{ac}A_{bd} \tag{A.35}$$

The Levi–Civita tensors are

$$\begin{aligned} \epsilon_{abc} &= (A)^{1/2}e_{abc} = \frac{1}{[1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})]^3}\, e_{abc} \\ \epsilon^{abc} &= \frac{1}{(A)^{1/2}}\, e^{abc} = [1 - \tfrac{1}{4}(\mathbf{v}\cdot\mathbf{v})]^3 e^{abc} \end{aligned} \tag{A.36}$$

where e^{abc} and e_{abc} are completely skew and $e^{123} = e_{123} = 1$.

Denote the covariant derivative by a semicolon, then

$$u^\mu_{,a\,;b} = \frac{\partial^2 u^\mu}{\partial v^a \partial v^b} - \left\{ \begin{matrix} c \\ a\ b \end{matrix} \right\}\frac{\partial u^\mu}{\partial v^c} = A_{ab}u^\mu \tag{A.37}$$

a relation known as Gauss' formula.

The momentum-energy gradient (A.20) has the three-dimensional form

$$\frac{\partial}{\partial p_\mu} = u^\mu(\mathbf{v})\frac{\partial}{\partial M} - \frac{1}{M}A^{ab}(\mathbf{v})u^\mu_{,a}(\mathbf{v})\frac{\partial}{\partial v^b} \tag{A.38}$$

Given a Lorentz transformation Λ, write

$$\mathbf{v}' = \Lambda^{-1}\mathbf{v} \tag{A.39}$$

where

$$u^\mu(\mathbf{v}') = \Lambda^{-1\mu}{}_\nu u^\nu(\mathbf{v}) \tag{A.40}$$

then

$$\begin{aligned} \Lambda^\mu{}_\nu u^\nu_{,b}(\mathbf{v}')\frac{\partial v^{b'}}{\partial v^a} &= u^\mu_{,a}(\mathbf{v}) \\ \frac{\partial v^{b'}}{\partial v^a} &= -u^\mu_{,a}(\mathbf{v})\Lambda_{\mu\nu}u^\nu_{,c}(\mathbf{v}')A^{cb}(\mathbf{v}') \end{aligned} \tag{A.41}$$

The second equation of (A.41) shows that the Jacobian is just the well-known Wigner rotation up to trivial stretch factors. From this fact, it follows that particle wave functions are just intrinsic tensors on the mass hyperboloid. Moreover, the correct momentum-energy gradient of such wave functions is given by applying (A.38), with $\partial/\partial v^b$ replaced by the covariant $\delta/\delta v^b$.

Finally, under a coordinate transformation, the Christoffel symbols transform according to

$$\begin{aligned} \frac{\partial^2 v^{a'}}{\partial v^b \partial v^c} &= \left\{ \begin{matrix} d \\ b\ c \end{matrix} \right\} \frac{\partial v^{a'}}{\partial v^d} - \left\{ \begin{matrix} a \\ d\ e \end{matrix} \right\}' \frac{\partial v^{d'}}{\partial v^b}\frac{\partial v^{e'}}{\partial v^c} \\ \frac{\partial^2 v^a}{\partial v^{b'} \partial v^{c'}} &= \left\{ \begin{matrix} d \\ b\ c \end{matrix} \right\}' \frac{\partial v^a}{\partial v^d} - \left\{ \begin{matrix} a \\ d\ e \end{matrix} \right\} \frac{\partial v^d}{\partial v^{b'}}\frac{\partial v^e}{\partial v^{c'}} \end{aligned} \tag{A.42}$$

APPENDIX B: THE HERMITICITY OF X^μ

First, consider the position-time operator defined in Section 2. Integrating the first term of (2.6) by parts, one obtains

$$\begin{aligned} -i\int dM\, d\mu(\mathbf{v})M^3 A^{ab}(\mathbf{v})\frac{\partial \psi^*_a(M,\mathbf{v})}{\partial M}u^\mu(\mathbf{v})\phi_b(M,\mathbf{v}) \\ = -i\int d\mu(\mathbf{v})M^3 A^{ab}(\mathbf{v})\psi^*_a(M,\mathbf{v})u^\mu(\mathbf{v})\phi_b(M,\mathbf{v})\Big|_{M=0}^{M=\infty} \\ + i\int d\mu(M,\mathbf{v})A^{ab}(\mathbf{v})\psi^*_a(M,\mathbf{v})u^\mu(\mathbf{v})\frac{\partial \phi_b(M,\mathbf{v})}{\partial M} \\ + 3i\int d\mu(M,\mathbf{v})\frac{1}{M}A^{ab}(\mathbf{v})\psi^*_a(M,\mathbf{v})u^\mu(\mathbf{v})\phi_b(M,\mathbf{v}) \end{aligned} \tag{B.1}$$

On a curved space, it is permissible to integrate by parts in the usual manner provided that the appropriate covariant quantities are employed.

Thus the second term of (2.6) gives

$$i\int M^3\,dM\,d\mu(\mathbf{v})\,\frac{1}{M}\,A^{ab}(\mathbf{v})A^{cd}(\mathbf{v})u^{\mu}_{,c}(\mathbf{v})\,\frac{\delta\psi^*_a(M,\mathbf{v})}{\delta v^d}\,\phi_b(M,\mathbf{v})$$

$$= i\int M^3\,dM\,d\mu(\mathbf{v})\,\frac{\delta}{\delta v^d}\left[\frac{1}{M}\,A^{ab}(\mathbf{v})A^{cd}(\mathbf{v})u^{\mu}_{,c}(\mathbf{v})\psi^*_a(M,\mathbf{v})\phi_b(M,\mathbf{v})\right]$$

$$- i\int d\mu(M,\mathbf{v})\,\frac{1}{M}\,A^{ab}(\mathbf{v})A^{cd}(\mathbf{v})\psi^*_a(M,\mathbf{v})u^{\mu}_{,c}(\mathbf{v})\,\frac{\delta\phi_b(M,\mathbf{v})}{\delta v^d}$$

$$- 3i\int d\mu(M,\mathbf{v})\,\frac{1}{M}\,A^{ab}(\mathbf{v})\psi^*_a(M,\mathbf{v})u^{\mu}(\mathbf{v})\phi_b(M,\mathbf{v}) \qquad \text{(B.2)}$$

where the last term comes from applying the relation (A.37) and observing that $A^{cd}(\mathbf{v})A_{cd}(\mathbf{v}) = 3$. The first term on the right-hand side of (B.2) leads to a surface integral at $|\mathbf{v}| = 2$ (infinite momentum) which is assumed to vanish. The integrated terms in (B.1) may also be neglected. It is assumed that the wave functions decrease sufficiently rapidly in modulus as $M \to \infty$. If (2.5) and (2.6) are well defined, then as $M \to 0$, the wave functions are not more singular than $1/M$ so that the integrated term for $M = 0$ must also vanish.

Combining (B.1) and (B.2), one readily obtains

$$(X^{\mu}\psi,\,\phi) = (\psi,\,X^{\mu}\phi) \qquad \text{(B.3)}$$

This derivation can, of course, be carried out using only the standard formula for integration by parts. Then, the work is most efficiently organized by expressing the partial derivative in terms of the covariant derivative and terms involving the Christoffel symbols. The numerous additional terms vanish in pairs.

It is evident that the position-time operator for the case of a given fixed mass is also Hermitian. Integrating the first term on the right-hand side of (3.3) by parts gives the result in (B.2) apart from a change in the measure throughout. The second term on the right-hand side of (3.3) combines with the last term of (B.2) to give the last term in (3.2).

APPENDIX C: THE COVARIANCE OF X^{μ}

For the operator X^{μ} given by (2.7),

$$(\psi,\,U^{-1}(\Lambda)X^{\mu}U(\Lambda)\phi)$$

$$= (U(\Lambda)\psi,\,X^{\mu}U(\Lambda)\phi)$$

$$= \int d\mu(M,\mathbf{v})A^{ab}(\mathbf{v})[U(\Lambda)\psi]^*_a(M,\mathbf{v})$$

$$\times i\left[u^{\mu}(\mathbf{v})\,\frac{\partial}{\partial M} - \frac{1}{M}\,A^{cd}(\mathbf{v})u^{\mu}_{,c}(\mathbf{v})\,\frac{\delta}{\delta v^d}\right][U(\Lambda)\phi]_b(M,\mathbf{v}) \qquad \text{(C.1)}$$

Using (2.2) and (A.40), one obtains

$$
\begin{aligned}
& i\int d\mu(M,\mathbf{v})A^{ab}(\mathbf{v})[U(\Lambda)\psi]^*_a(M,\mathbf{v})u^\mu(\mathbf{v})\frac{\partial}{\partial M}[U(\Lambda)\phi]_b(M,\mathbf{v}) \\
&\quad = i\int d\mu(M,\mathbf{v})A^{ab}(\mathbf{v})\frac{\partial(\Lambda^{-1}\mathbf{v})^c}{\partial v^a}\psi^*_c(M,\Lambda^{-1}\mathbf{v}) \\
&\qquad \times \Lambda^\mu{}_\nu u^\nu(\Lambda^{-1}\mathbf{v})\frac{\partial(\Lambda^{-1}\mathbf{v})^d}{\partial v^b}\frac{\partial\phi_d}{\partial M}(M,\Lambda^{-1}\mathbf{v}) \qquad \text{(C.2)} \\
&\quad = i\int d\mu(M,\Lambda^{-1}\mathbf{v})A^{ab}(\Lambda^{-1}\mathbf{v})\psi^*_a(M,\Lambda^{-1}\mathbf{v})\Lambda^\mu{}_\nu u^\nu(\Lambda^{-1}\mathbf{v})\frac{\partial\phi_b}{\partial M}(M,\Lambda^{-1}\mathbf{v}) \\
&\quad = \Lambda^\mu{}_\nu\, i\int d\mu(M,\mathbf{v})A^{ab}(\mathbf{v})\psi^*_a(M,\mathbf{v})u^\nu(\mathbf{v})\frac{\partial\phi_b}{\partial M}(M,\mathbf{v})
\end{aligned}
$$

Using (2.2), (A.41), and (A.42), one obtains

$$
\begin{aligned}
& -i\int d\mu(M,\mathbf{v})A^{ab}(\mathbf{v})[U(\Lambda)\psi]^*_a(M,\mathbf{v})\frac{1}{M}A^{cd}(\mathbf{v})u^\mu_{,c}(\mathbf{v}) \\
&\qquad\qquad \times\left[\frac{\partial}{\partial v^d}[U(\Lambda)\phi]_b(M,\mathbf{v}) - \begin{Bmatrix} f \\ d\ b \end{Bmatrix}[U(\Lambda)\phi]_f(M,\mathbf{v})\right] \\
&\quad = -i\int d\mu(M,\Lambda^{-1}\mathbf{v})A^{ab}(\mathbf{v})\frac{\partial(\Lambda^{-1}\mathbf{v})^e}{\partial v^a}\psi^*_e(M,\Lambda^{-1}\mathbf{v}) \\
&\qquad \times\frac{1}{M}A^{cd}(\mathbf{v})\Lambda^\mu{}_\nu u^\nu_{,h}(\Lambda^{-1}\mathbf{v})\frac{\partial(\Lambda^{-1}\mathbf{v})^h}{\partial v^c}\left[\frac{\partial(\Lambda^{-1}\mathbf{v})^f}{\partial v^b}\frac{\partial(\Lambda^{-1}\mathbf{v})^g}{\partial v^d}\frac{\partial\phi_f(M,\Lambda^{-1}\mathbf{v})}{\partial(\Lambda^{-1}\mathbf{v})^g}\right. \\
&\qquad \left. + \frac{\partial^2(\Lambda^{-1}\mathbf{v})^f}{\partial v^d\partial v^b}\phi_f(M,\Lambda^{-1}\mathbf{v}) - \begin{Bmatrix} f \\ d\ b \end{Bmatrix}\frac{\partial(\Lambda^{-1}\mathbf{v})^g}{\partial v^f}\phi_g(M,\Lambda^{-1}\mathbf{v})\right] \\
&\quad = -i\int d\mu(M,\Lambda^{-1}\mathbf{v})A^{ab}(\mathbf{v})\frac{\partial(\Lambda^{-1}\mathbf{v})^e}{\partial v^a}\psi^*_e(M,\Lambda^{-1}\mathbf{v}) \qquad \text{(C.3)} \\
&\qquad \times\frac{1}{M}A^{cd}(\mathbf{v})\Lambda^\mu{}_\nu u^\nu_{,h}(\Lambda^{-1}\mathbf{v})\frac{\partial(\Lambda^{-1}\mathbf{v})^h}{\partial v^c}\left[\frac{\partial(\Lambda^{-1}\mathbf{v})^f}{\partial v^b}\frac{\partial(\Lambda^{-1}\mathbf{v})^g}{\partial v^d}\frac{\partial\phi_f(M,\Lambda^{-1}\mathbf{v})}{\partial(\Lambda^{-1}\mathbf{v})^g}\right. \\
&\qquad \left. - \begin{Bmatrix} f \\ k\ l \end{Bmatrix}'\frac{\partial(\Lambda^{-1}\mathbf{v})^k}{\partial v^d}\frac{\partial(\Lambda^{-1}\mathbf{v})^l}{\partial v^b}\phi_f(M,\Lambda^{-1}\mathbf{v})\right] \\
&\quad = -i\Lambda^\mu{}_\nu\int d\mu(M,\Lambda^{-1}\mathbf{v})A^{ab}(\Lambda^{-1}\mathbf{v})\psi^*_a(M,\Lambda^{-1}\mathbf{v}) \\
&\qquad \times\frac{1}{M}A^{cd}(\Lambda^{-1}\mathbf{v})u^\nu_{,c}(\Lambda^{-1}\mathbf{v})\left[\frac{\partial\phi_b(M,\Lambda^{-1}\mathbf{v})}{\partial(\Lambda^{-1}\mathbf{v})^d} - \begin{Bmatrix} e \\ b\ d \end{Bmatrix}'\phi_e(M,\Lambda^{-1}\mathbf{v})\right] \\
&\quad = -i\Lambda^\mu{}_\nu\int d\mu(M,\mathbf{v})A^{ab}(\mathbf{v})\psi^*_a(M,\mathbf{v})\frac{1}{M}A^{cd}(\mathbf{v})u^\nu_{,c}(\mathbf{v})\frac{\delta}{\delta v^d}\phi_b(M,\mathbf{v})
\end{aligned}
$$

Hence

$$(\psi, U^{-1}(\Lambda)X^{\mu}U(\Lambda)\phi) = \Lambda^{\mu}{}_{\nu}(\psi, X^{\nu}\phi) \tag{C.4}$$

The result (C.4) is also valid for the operator X^{μ} defined by (3.4) because the transformation of the first term follows the same pattern as in (C.3), while the transformation of the second term follows the same pattern as in (C.2).

APPENDIX D: PURE LORENTZ TRANSFORMATIONS AND PARALLEL TRANSPORT ALONG MASS HYPERBOLOID GEODESICS

Given two points U_1 and U_2 on the forward unit mass hyperboloid with coordinates $\mathbf{v}_1 = \mathbf{v}(U_1)$ and $\mathbf{v}_2 = \mathbf{v}(U_2)$, respectively, denote the pure Lorentz transformation in the plane spanned by $u^{\mu}(\mathbf{v}_1)$ and $u^{\mu}(\mathbf{v}_2)$ which takes $u^{\mu}(\mathbf{v}_1)$ into $u^{\mu}(\mathbf{v}_2)$ by $B(\mathbf{v}_2, \mathbf{v}_1)$. Then

$$\Lambda B(\mathbf{v}_2, \mathbf{v}_1)\Lambda^{-1} = B(\Lambda\mathbf{v}_2, \Lambda\mathbf{v}_1) \tag{D.1}$$

If $\mathbf{v} = \mathbf{v}(U)$ are the coordinates of a point U between U_1 and U_2 in the plane spanned by $u^{\mu}(\mathbf{v}_1)$ and $u^{\mu}(\mathbf{v}_2)$, then the point U traces out the geodesic between U_1 and U_2 as the hyperbolic angle ξ given by

$$\cosh \xi = g_{\mu\nu}u^{\mu}(\mathbf{v})u^{\nu}(\mathbf{v}_1) \tag{D.2}$$

increases from $\xi = 0$ to $\xi = \xi_{12}$, where

$$\cosh \xi_{12} = g_{\mu\nu}u^{\mu}(\mathbf{v}_2)u^{\nu}(\mathbf{v}_1) \tag{D.3}$$

Moreover, ξ is the length along the geodesic from U_1 to U. (Compare with the analogous case on a sphere.)

The Wigner rotation corresponding to the Lorentz transformation $B(\mathbf{v}_2, \mathbf{v}_1)$ is given by

$$R(\mathbf{v}_2, \mathbf{v}_1) = B(0, \mathbf{v}_2)B(\mathbf{v}_2, \mathbf{v}_1)B(\mathbf{v}_1, 0) \tag{D.4}$$

The task is to evaluate $R(\mathbf{v} + d\mathbf{v}, \mathbf{v})$ to first order and show that (4.13) and (4.12) agree to first order.

Using (A.37), one obtains to the order indicated

$$\begin{aligned} u^{\mu}(\mathbf{v} + d\mathbf{v}) &= u^{\mu}(\mathbf{v})\left[1 + \frac{1}{2}(dv)^2\right] + u^{\mu}_{,a}(\mathbf{v})\, dv^a + \frac{1}{2}\begin{Bmatrix} c \\ a\ b \end{Bmatrix} u^{\mu}_{,c}(\mathbf{v})\, dv^a\, dv^b \\ u^{\mu}_{,a}(\mathbf{v} + d\mathbf{v}) &= u^{\mu}_{,a}(\mathbf{v}) + A_{ab}(\mathbf{v})u^{\mu}(\mathbf{v})\, dv^b + \begin{Bmatrix} c \\ a\ b \end{Bmatrix} u^{\mu}_{,c}(\mathbf{v})\, dv^b \end{aligned} \tag{D.5}$$

where

$$(dv)^2 = A_{ab}(\mathbf{v})\, dv^a\, dv^b \tag{D.6}$$

Thus

$$\begin{aligned}\cosh(d\xi) &= g_{\mu\nu}u^{\mu}(\mathbf{v}+d\mathbf{v})u^{\nu}(\mathbf{v}) = 1+\tfrac{1}{2}(dv)^2\\ \sinh(d\xi) &= d\xi = dv\end{aligned} \tag{D.7}$$

The pure Lorentz transformation which takes $\mathbf{v}$ into $\mathbf{v}+d\mathbf{v}$ is given by

$$\begin{aligned}B(\mathbf{v}+d\mathbf{v},\mathbf{v})^{\mu\nu} &= g^{\mu\nu} + dv\left[u^{\mu}_{\cdot c}(\mathbf{v})\frac{dv^c}{dv}u^{\nu}(\mathbf{v}) - u^{\nu}_{\cdot c}(\mathbf{v})\frac{dv^c}{dv}u^{\mu}(\mathbf{v})\right]\\ &= g^{\mu\nu} + \left[n_c{}^{\mu}(\mathbf{v})\frac{dv^c}{1-\frac{1}{4}(\mathbf{v}\cdot\mathbf{v})}u^{\nu}(\mathbf{v}) - n_c{}^{\nu}(\mathbf{v})\frac{dv^c}{1-\frac{1}{4}(\mathbf{v}\cdot\mathbf{v})}u^{\mu}(\mathbf{v})\right]\end{aligned} \tag{D.8}$$

From the second equation of (D.5), one obtains

$$\begin{aligned}n_a{}^{\mu}(\mathbf{v}+d\mathbf{v}) &= n_a{}^{\mu}(\mathbf{v}) + u^{\mu}(\mathbf{v})\frac{dv^a}{1-\frac{1}{4}(\mathbf{v}\cdot\mathbf{v})}\\ &\quad + \frac{1}{2}\left[v^a n_c{}^{\mu}(\mathbf{v})\frac{dv^c}{1-\frac{1}{4}(\mathbf{v}\cdot\mathbf{v})} - v^c n_c{}^{\mu}(\mathbf{v})\frac{dv^a}{1-\frac{1}{4}(\mathbf{v}\cdot\mathbf{v})}\right]\end{aligned} \tag{D.9}$$

Then the required Wigner rotation is given by [cf. (A.41)]

$$R(\mathbf{v}+d\mathbf{v},\mathbf{v})_{ab} = -n_a{}^{\mu}(\mathbf{v}+d\mathbf{v})B(\mathbf{v}+d\mathbf{v},\mathbf{v})_{\mu\nu}n_b{}^{\nu}(\mathbf{v}) \tag{D.10}$$

To first order, one obtains

$$R(\mathbf{v}+d\mathbf{v},\mathbf{v})_{ab} = \delta_{ab} + \frac{1}{2}\left[v^a\frac{dv^b}{1-\frac{1}{4}(\mathbf{v}\cdot\mathbf{v})} - v^b\frac{dv^a}{1-\frac{1}{4}(\mathbf{v}\cdot\mathbf{v})}\right] \tag{D.11}$$

Since $R(\mathbf{v},\mathbf{v}+d\mathbf{v})_{ab} = R(\mathbf{v}+d\mathbf{v},\mathbf{v})_{ba}$, it follows that (4.13) gives (4.12) to first order.

Define the 3×3 rotation matrices S_{ab}

$$(S_{ab})_{cd} = -i(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) \tag{D.12}$$

which satisfy

$$[S_{ab}, S_{cd}] = -i(\delta_{ac}S_{bd} + \delta_{bd}S_{ac} - \delta_{ad}S_{bc} - \delta_{bc}S_{ad}) \tag{D.13}$$

Then

$$\begin{aligned}R(\mathbf{v},\mathbf{v}+d\mathbf{v}) &= I + \tfrac{1}{2}\Omega_{ab}S_{ab}\\ \Omega_{ab} &= -\frac{1}{2}\left[\frac{v^a\,dv^b - v^b\,dv^a}{1-\frac{1}{4}(\mathbf{v}\cdot\mathbf{v})}\right]\end{aligned} \tag{D.14}$$

Finally, the following expressions for the parallel transport of a Wigner vector through a finite distance along a geodesic are given without proof:

$$\psi^{(w)}_a(\mathbf{v}, \parallel \to \mathbf{v}_2) = R(\mathbf{v}_2,\mathbf{v}_1)_{ab}\psi^{(w)}_b(\mathbf{v}_1) \tag{D.15}$$

The rotation $R(\mathbf{v}_2, \mathbf{v}_1)$ defined by (D.4) is explicitly given by

$$R(\mathbf{v}_2, \mathbf{v}_1)_{ab} = \delta_{ab} - \frac{2}{[1 - \frac{1}{4}(\mathbf{v}_2\cdot\mathbf{v}_1)]^2 + \frac{1}{16}[(\mathbf{v}_2 \times \mathbf{v}_1)\cdot(\mathbf{v}_2 \times \mathbf{v}_1)]}$$
$$\times \{[1 - \tfrac{1}{4}(\mathbf{v}_2\cdot\mathbf{v}_1)]\tfrac{1}{4}(v_2{}^a v_1{}^b - v_2{}^b v_1{}^a) + \tfrac{1}{4}(\mathbf{v}_2\cdot\mathbf{v}_2)\tfrac{1}{4}v_1{}^a v_1{}^b$$
$$+ \tfrac{1}{4}(\mathbf{v}_1\cdot\mathbf{v}_1)\tfrac{1}{4}v_2{}^a v_2{}^b - \tfrac{1}{4}(\mathbf{v}_2\cdot\mathbf{v}_1)\tfrac{1}{4}(v_2{}^a v_1{}^b + v_2{}^b v_1{}^a)\} \tag{D.16}$$

That $\psi_a^{(w)}(\mathbf{v}, \| \to \mathbf{v}_2)$ is a vector at v_2 follows from

$$\begin{aligned}
&[U(\Lambda)\psi]_a^{(w)}(\mathbf{v}, \| \to \mathbf{v}_2) \\
&\quad = R(\mathbf{v}_2, \mathbf{v}_1)_{ab}[U(\Lambda)\psi]_b^{(w)}(\mathbf{v}_1) \\
&\quad = [B^{-1}(\mathbf{v}_2)B(\mathbf{v}_2, \mathbf{v}_1)B(\mathbf{v}_1)B^{-1}(\mathbf{v}_1)\Lambda B(\Lambda^{-1}\mathbf{v}_1)]_{ab}\psi_b^{(w)}(\Lambda^{-1}\mathbf{v}_1) \qquad \text{(D.17)} \\
&\quad = [B^{-1}(\mathbf{v}_2)\Lambda B(\Lambda^{-1}\mathbf{v}_2)B^{-1}(\Lambda^{-1}\mathbf{v}_2)B(\Lambda^{-1}\mathbf{v}_2, \Lambda^{-1}\mathbf{v}_1)B(\Lambda^{-1}\mathbf{v}_1)]_{ab}\psi_b^{(w)}(\Lambda^{-1}\mathbf{v}_1) \\
&\quad = [B^{-1}(\mathbf{v}_2)\Lambda B^{-1}(\Lambda^{-1}\mathbf{v}_2)]_{ab}\psi_b^{(w)}(\Lambda^{-1}\mathbf{v}_1\| \to \Lambda^{-1}\mathbf{v}_2)
\end{aligned}$$

APPENDIX E: EVALUATION OF $J^{\mu\nu}$

For an infinitesimal Lorentz transformation

$$\Lambda^{-1\mu\nu} = g^{\mu\nu} - \omega^{\mu\nu} \tag{E.1}$$

set

$$\Lambda^{-1}\mathbf{v} = \mathbf{v} + \Delta\mathbf{v} \tag{E.2}$$

Then, the first-order terms of the first equation in (D.5) together with (A.40) gives

$$\Delta v^a = \tfrac{1}{2}\omega_{\mu\nu}A^{ab}(\mathbf{v})(u^\mu{}_{,b}(\mathbf{v})u^\nu(\mathbf{v}) - u^\nu{}_{,b}(\mathbf{v})u^\mu(\mathbf{v})) \tag{E.3}$$

Since the covariant derivative of $A^{ab}(\mathbf{v})$ is zero,

$$A^{ab}(\mathbf{v} + \Delta\mathbf{v}) = A^{ab}(\mathbf{v}) - A^{bd}(\mathbf{v})\left\{\begin{matrix} a \\ d\ c \end{matrix}\right\}\Delta v^c - A^{ad}(\mathbf{v})\left\{\begin{matrix} b \\ d\ c \end{matrix}\right\}\Delta v^c \tag{E.4}$$

Substituting (E.4) and the second of the equations (D.5) into (A.41), one obtains to first order

$$\frac{\partial(\Lambda^{-1}\mathbf{v})^b}{\partial v^a} = \delta_a{}^b - \tfrac{1}{2}\omega_{\mu\nu}(u^\mu{}_{,a}(\mathbf{v})u^\nu{}_{,c}(\mathbf{v}) - u^\nu{}_{,a}(\mathbf{v})u^\mu{}_{,c}(\mathbf{v}))A^{cb}(\mathbf{v}) - \left\{\begin{matrix} b \\ a\ f \end{matrix}\right\}\Delta v^f \tag{E.5}$$

Also

$$\psi_b(\Lambda^{-1}\mathbf{v}) = \psi_b(\mathbf{v}) + \frac{\partial\psi_b(\mathbf{v})}{\partial c^c}\Delta v^c \tag{E.6}$$

Using (5.1) for infinitesimal ω and (E.5) and (E.6) in (2.2), one obtains

$$(J^{\mu\nu})_{ab} = A_{ab}(\mathbf{v})iA^{cd}(\mathbf{v})[u^{\mu}(\mathbf{v})u^{\nu}_{,c}(\mathbf{v}) - u^{\nu}(\mathbf{v})u^{\mu}_{,c}(\mathbf{v})]\frac{\delta}{\delta v^d}$$

$$+ i[u^{\mu}_{,a}(\mathbf{v})u^{\nu}_{,b}(\mathbf{v}) - u^{\nu}_{,a}(\mathbf{v})u^{\mu}_{,b}(\mathbf{v})] \tag{E.7}$$

or

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu} \tag{E.8}$$

where $L^{\mu\nu}$ and $S^{\mu\nu}$ are given by (2.16) and (2.15), respectively.

For the case of the more general representation given by (4.20), first evaluate the Wigner rotation $[B^{-1}(\mathbf{v})\Lambda B(\Lambda^{-1}\mathbf{v})]$:

$$\begin{aligned}[B^{-1}(\mathbf{v})\Lambda B(\Lambda^{-1}\mathbf{v})]_{ab} &= -n_a{}^{\mu}(\mathbf{v})\Lambda_{\mu\nu}n_b{}^{\nu}(\mathbf{v}+\Delta\mathbf{v}) \\ &= \delta_{ab} - \tfrac{1}{2}\omega_{\mu\nu}(n_a{}^{\mu}n_b{}^{\nu} - n_a{}^{\nu}n_b{}^{\mu}) - \frac{1}{2}\left[\frac{v^a\Delta v^b - v^b\Delta v^a}{1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})}\right]\end{aligned} \tag{E.9}$$

Then, the infinitesimal form of (4.21) gives

$$D^{(s)}_{\lambda\mu}[B^{-1}(\mathbf{v})\Lambda B(\Lambda^{-1}\mathbf{v})] = \delta_{\lambda\mu} - \frac{i}{4}\omega_{\rho\sigma}(n_a{}^{\rho}n_b{}^{\sigma} - n_a{}^{\sigma}n_b{}^{\rho})(S_{ab})_{\lambda\mu}$$

$$- \frac{i}{4}\left[\frac{v^a\Delta v^b - v^b\Delta v^a}{1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})}\right](S_{ab})_{\lambda\mu} \tag{E.10}$$

Using

$$\psi^{(w)}_{\mu}(\Lambda^{-1}\mathbf{v}) = \psi^{(w)}_{\mu}(\mathbf{v}) + \frac{\partial\psi^{(w)}_{\mu}(\mathbf{v})}{\partial v^c}\Delta v^c \tag{E.11}$$

along with (E.10), (4.24) and the infinitesimal form of (5.1), one obtains from (4.20) the result (E.8), where $L^{\mu\nu}$ is given by (4.25) and $S^{\rho\sigma}$ is given by

$$(S^{\rho\sigma})_{\lambda\mu} = \tfrac{1}{2}(n_a{}^{\rho}n_b{}^{\sigma} - n_a{}^{\sigma}n_b{}^{\rho})(S_{ab})_{\lambda\mu} \tag{E.12}$$

APPENDIX F: SPIN WAVE FUNCTIONS AND PROJECTION OPERATORS

As mentioned in Section 6, the unitary irreducible representations of the homogeneous Lorentz group 0(1, 3) (Naimark, 1964) may be realized as unitary operators $U(\Lambda)$: $\mathscr{H} \to \mathscr{H}$, where $\Lambda \in 0(1, 3)$ and $\mathscr{H}$ is a Hilbert space with a basis $\{|jm\rangle\}$ satisfying

$$\langle j'm'|jm\rangle = \delta_{j'j}\delta_{m'm} \tag{F.1}$$

The range of the indices j, m is given by (6.3). If $B(\mathbf{v}) \in 0(1, 3)$ is the pure

Lorentz transformation which takes $u^\mu(\mathbf{0})$ into $u^\mu(\mathbf{v})$, then the functions $B_{jm}(\mathbf{v}, s, \lambda)$ are defined by

$$B_{jm}(\mathbf{v}, s, \lambda) = \langle jm|U(B(\mathbf{v}))|s\lambda\rangle \tag{F.2}$$

Since

$$\begin{aligned} U_{jm;j'm'}(\Lambda)B_{j'm'}(\Delta^{-1}\mathbf{v}, s, \lambda) &= U_{jm;j'm'}(\Lambda)\langle j'm'|U(B(\Lambda^{-1}\mathbf{v}))|s\lambda\rangle \\ &= \langle jm|U(\Lambda)U(B(\Lambda^{-1}\mathbf{v}))|s\lambda\rangle \\ &= \langle jm|U(B(\mathbf{v}))U(B^{-1}(\mathbf{v})\Lambda B(\Lambda^{-1}\mathbf{v}))|s, \lambda\rangle \\ &= B_{jm}(\mathbf{v}, s, \lambda')D^{(s)}_{\lambda'\lambda}(B^{-1}(\mathbf{v})\Lambda B(\Lambda^{-1}\mathbf{v})) \end{aligned} \tag{F.3}$$

one has the relation

$$U_{jm;j'm'}(\Lambda)B_{j'm'}(\Lambda^{-1}\mathbf{v}, s, \lambda) = B_{jm}(\mathbf{v}, s, \lambda)D^{(s)}_{\lambda'\lambda}(B^{-1}(\mathbf{v})\Lambda B(\Lambda^{-1}\mathbf{v})) \tag{F.4}$$

which is useful for discussing the transformation properties of the wave functions. From the unitarity of the representation, it follows that

$$\begin{aligned} \sum_{jm} B^*_{jm}(\mathbf{v}, s', \lambda')B_{jm}(\mathbf{v}, s, \lambda) &= \delta_{s's}\delta_{\lambda'\lambda} \\ \sum_{s\lambda} B_{j'm'}(\mathbf{v}, s, \lambda)B^*_{jm}(\mathbf{v}, s, \lambda) &= \delta_{j'j}\delta_{m'm} \end{aligned} \tag{F.5}$$

The relation

$$U^{-1}_{jm,j'm'}(\Lambda)B_{j'm'}(\Lambda\mathbf{v}, s, \lambda) = B_{jm}(\mathbf{v}, s, \lambda')D^{(s)}_{\lambda'\lambda}(B^{-1}(\mathbf{v})\Lambda^{-1}B(\Lambda\mathbf{v})) \tag{F.6}$$

is useful in the discussion of the transformation properties of the field operators. Its demonstration is similar to that given in (F.3).

The projection operator onto a subspace of given spin s is

$$\mathrm{Pr}^{(s)}(\mathbf{v})_{j'm';jm} = \sum_{\lambda} B_{j'm'}(\mathbf{v}, s, \lambda)B^*_{jm}(\mathbf{v}, s, \lambda) \tag{F.7}$$

Since an infinite-dimensional representation of 0(1, 3) is being used, the explicit expressions for these projection operators are more complicated than those obtained in the finite-dimensional spinor formalism. They can, however, be expressed conveniently by means of infinite products. Define

$$S_i(\mathbf{v}) = -\frac{1}{M}W_\mu n_i{}^\mu(\mathbf{v}) \tag{F.8}$$

where W_μ is given by (5.14). Then

$$[S_i(\mathbf{v}), S_j(\mathbf{v})] = ie_{ijk}S_k(\mathbf{v}) \tag{F.9}$$

and

$$\sum_i S_i(\mathbf{v})S_i(\mathbf{v}) = -\frac{1}{M^2}W_\mu W^\mu \tag{F.10}$$

and

$$\begin{aligned} \mathbf{S}^2(\mathbf{v})_{jm;j'm'}B_{j'm'}(\mathbf{v}, s, \lambda) &= s(s+1)B_{jm}(\mathbf{v}, s, \lambda) \\ S_3(\mathbf{v})_{jm;j'm'}B_{j'm'}(\mathbf{v}, s, \lambda) &= \lambda B_{jm}(\mathbf{v}, s, \lambda) \end{aligned} \tag{F.11}$$

In terms of the generators $S^{\mu\nu}$, one has

$$\mathbf{S}^2(\mathbf{v}) = \tfrac{1}{2}g_{\mu\nu}g_{\kappa\lambda}S^{\mu\kappa}S^{\nu\lambda} - g_{\mu\nu}g_{\kappa\lambda}S^{\mu\kappa}u^{\lambda}(\mathbf{v})g_{\rho\sigma}S^{\nu\rho}u^{\sigma}(\mathbf{v}) \tag{F.12}$$

For $s \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}$,

$$\mathrm{Pr}^{(s)}(\mathbf{v}) = \prod_{j \neq s}\left[1 - \frac{\mathbf{S}^2(\mathbf{v})}{j(j+1)}\right] \Big/ \prod_{j \neq s}\left[1 - \frac{s(s+1)}{j(j+1)}\right] \tag{F.13}$$

For $s = 0$,

$$\mathrm{Pr}^{(0)}(\mathbf{v}) = \prod_{j=1}^{\infty}\left[1 - \frac{\mathbf{S}^2(\mathbf{v})}{j(j+1)}\right] \tag{F.14}$$

and for $s \in \{1, 2, 3, \ldots\}$

$$\mathrm{Pr}^{(s)}(\mathbf{v}) = \frac{\mathbf{S}^2(\mathbf{v})}{s(s+1)}\left\{\prod_{j \neq 0,s}\left[1 - \frac{\mathbf{S}^2(\mathbf{v})}{j(j+1)}\right] \Big/ \prod_{j \neq 0,s}\left[1 - \frac{s(s+1)}{j(j+1)}\right]\right\} \tag{F.15}$$

The infinite products in (F.13), (F.14), and (F.15) are absolutely convergent since the series $\sum_j 1/[j(j+1)]$ is absolutely convergent.

APPENDIX G: LIGHTLIKE AND SPACELIKE MOMENTA

The results of Sections 2–5 may be extended to the case of lightlike and spacelike momenta once a covariant momentum derivative for intrinsic tensors on the appropriate mass shells is defined.

For the spacelike case, one may introduce the orthogonal curvilinear coordinates (s, ϕ, θ) by

$$p^{\mu} = M(sh\ s,\ ch\ s \sin\theta,\ ch\ s \cos\phi \sin\theta,\ ch\ s \cos\phi \cos\theta) \tag{G.1}$$

Then the Riemann metric on the mass shell is given by

$$\frac{1}{M^2}\, g_{\mu\nu}\, dp^{\mu}\, dp^{\nu} = ds^2 - ch^2 s\, d\theta^2 - ch^2 s \cos^2\phi\, d\theta^2 \tag{G.2}$$

and the unit tangent vectors are

$$\begin{aligned} n_s{}^{\mu} &= (ch\ s,\ sh\ s \sin\phi,\ sh\ s \cos\phi \sin\theta,\ sh\ s \cos\phi \cos\theta) \\ n_{\phi}{}^{\mu} &= (0,\ \cos\phi,\ -\sin\phi \sin\theta,\ -\sin\phi \cos\theta) \\ n_{\theta}{}^{\mu} &= (0,\ 0,\ \cos\theta,\ -\sin\theta) \end{aligned} \tag{G.3}$$

Given any two points A and B on the mass shell with coordinates $(s(A), \phi(A), \theta(A))$ and $(s(B), \phi(B), \theta(B))$, one may take the vectors (G.3) for the point A and parallel transport them along the geodesic from A to B and compare the resulting unit vectors with the vectors (G.3) for the point B. The two sets of unit vectors at the point B are related by an element of the group 0(1, 2). This group element defines the transformation that must be applied to any intrinsic tensor at A in order to obtain the tensor parallel transported to B along the geodesic connecting A and B. In the present case, since a unitary spin basis is desired, the indices on the momentum wavefunction label the basis for a unitary irreducible representation of the group 0(1, 2) and the required transformation is the representation of the above group element relative to this infinite component basis. The covariant momentum derivative is defined in terms of parallel transport in the standard way. It is interesting to note that covariant differentiation of infinite component fields on de Sitter space–time may be defined in precisely the same way [with the groups 0(1, 4) and 0(1, 3) replacing the groups 0(1, 3) and 0(1, 2), respectively].

Unfortunately, the case of lightlike momentum cannot be treated in the same way because the intrinsic metric of the (forward) light cone is degenerate. Introducing the orthogonal curvilinear coordinates (ω, ξ^1, ξ^2) by

$$p^\mu = \omega\left[1, \frac{1 - \frac{1}{4}(\xi\cdot\xi)}{1 + \frac{1}{4}(\xi\cdot\xi)}, \frac{\xi^1}{1 + \frac{1}{4}(\xi\cdot\xi)}, \frac{\xi^2}{1 + \frac{1}{4}(\xi\cdot\xi)}\right] \tag{G.4}$$

one obtains for the metric

$$g_{\mu\nu}\, dp^\mu\, dp^\nu = -\frac{\omega^2}{[1 + \frac{1}{4}(\xi\cdot\xi)]^2}\, d\xi\cdot d\xi \tag{G.5}$$

Together with $p^\mu(\omega, \xi)$, the vectors

$$\begin{aligned} q^\mu(\omega, \xi) &= \frac{1}{\omega}\left[1, -\frac{1 - \frac{1}{4}(\xi\cdot\xi)}{1 + \frac{1}{4}(\xi\cdot\xi)}, \frac{-\xi^1}{1 + \frac{1}{4}(\xi\cdot\xi)}, \frac{-\xi^2}{1 + \frac{1}{4}(\xi\cdot\xi)}\right] \\ n_1{}^\mu(\xi) &= \frac{1}{1 + \frac{1}{4}(\xi\cdot\xi)}\left(0, -\xi^1, 1 + \frac{\xi\cdot\xi}{4} - \frac{\xi^1\xi^2}{2}, -\frac{\xi^1\xi^2}{2}\right) \\ n_2{}^\mu(\xi) &= \frac{1}{1 + \frac{1}{4}(\xi\cdot\xi)}\left(0, -\xi^2, -\frac{\xi^2\xi^1}{2}, 1 + \frac{\xi\cdot\xi}{4} - \frac{\xi^2\xi^2}{2}\right) \end{aligned} \tag{G.6}$$

form a complete set which satisfy

$$\begin{aligned} p\cdot p &= 0 = q\cdot q \\ p\cdot n_r &= 0 = q\cdot n_r \\ p\cdot q &= 2 \qquad n_r\cdot n_s = -\delta_{rs} \end{aligned} \tag{G.7}$$

Under a Lorentz transformation, the vectors $\Lambda q(\omega, \xi)$ and $\Lambda n_r(\xi)$ are related to the vectors $q(\Lambda\omega, \Lambda\xi)$ and $n_r(\Lambda\xi)$ by an element of the group $E(2)$. In order to define a covariant derivative, one must provide a standard *Lorentz invariant* rule for transporting the vectors (G.6) for a point A to any other point B on the forward light cone. At present, this problem is unsolved. However, once the decision to use infinite component fields has been made, a zero mass particle may be described by demanding that the wave function satisfy an additional equation of motion which expresses the proportionality between the Pauli–Lubanski operator and the four-momentum operator.

APPENDIX H: THOMAS PRECESSION

It was remarked in Section 10 that the expression for Thomas precession may be readily derived from equation (10.9)

$$\frac{d\mathscr{S}^a}{d\mathscr{T}} + \begin{Bmatrix} a \\ b\ c \end{Bmatrix} \mathscr{S}^b \frac{dv^c}{d\mathscr{T}} = 0 \tag{H.1}$$

First, it is necessary to convert from the basis $u^\mu_{,a}(\mathbf{v})$ to the normalized basis $n_a{}^\mu(\mathbf{v})$ defined by (4.3) by setting

$$\mathscr{S}^a = \left(1 - \frac{\mathbf{v}\cdot\mathbf{v}}{4}\right)\Sigma^a \tag{H.2}$$

The removal of this stretch factor results in the cancellation of one of the three terms in the Christoffel symbol of (H.1). One obtains

$$\frac{d\Sigma^a}{d\mathscr{T}} - \Omega^{ab}\Sigma^b = 0 \tag{H.3}$$

where

$$\Omega^{ab} = \frac{1}{2}\frac{1}{1 - \frac{1}{4}(\mathbf{v}\cdot\mathbf{v})}\left(v^a \frac{dv^b}{d\mathscr{T}} - v^b \frac{dv^a}{d\mathscr{T}}\right) \tag{H.4}$$

Equation (H.3) states that the spin vector precesses with the angular velocity given by (H.4).

It is customary to employ an alternate parametrization for the four velocity, namely,

$$\frac{dz^u}{d\mathscr{T}} = \frac{1}{(1 - \boldsymbol{\beta}\cdot\boldsymbol{\beta})^{1/2}}(1, \boldsymbol{\beta}) \tag{H.5}$$

Then,

$$\mathbf{v} = \frac{2\gamma}{\gamma + 1}\boldsymbol{\beta}$$
$$\gamma = \frac{1}{(1 - \boldsymbol{\beta}\cdot\boldsymbol{\beta})^{1/2}} \tag{H.6}$$

and

$$\Omega^{ab} = \frac{\gamma}{\gamma + 1}\left(\beta^a \frac{d\beta^b}{dt} - \beta^b \frac{d\beta^a}{dt}\right) \tag{H.7}$$

where t denotes the time in the laboratory frame and

$$d\mathscr{T} = \gamma\, dt \tag{H.8}$$

For a particle moving in a circle of radius r in the x-y plane of the laboratory frame with constant angular velocity ω,

$$\begin{aligned} x &= r\cos\omega t \\ y &= r\sin\omega t \\ z &= 0 \end{aligned} \tag{H.9}$$

The only nonzero components of Ω^{ab} are

$$\Omega^{12} = -\Omega^{21} = \frac{\gamma - 1}{\gamma}\omega \tag{H.10}$$

If laboratory time is used

$$\frac{d\Sigma^a}{dt} - \gamma\Omega^{ab}\Sigma^b = 0 \tag{H.11}$$

so that the precession rate is given by

$$\gamma\Omega^{12} = (\gamma - 1)\omega \tag{H.12}$$

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